Calculus 1, Midterm test 1, A

20th October, 2025

1. (8+12 points)

a) Let $z = 1 + 2i$. Calculate $\frac{z+6}{3z+\overline{z}}$ and give the result in algebraic form.

b) $\sqrt[3]{8i} = ?$ Give the roots in trigonometric form and also plot them.

2. (16 points) Let $a_n = \frac{6n^3 - n + 10}{2n^3 + 5n^2 + 8}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.01$.

3. (12 points) Find the limit of the following sequence: $a_n = n^2\left(\sqrt{n^6 - 7n} - \sqrt{n^6 + 3n}\right)$.

4. (14 points) Find the limit of the following sequence: $a_n = \sqrt[n]{\frac{2^n + n^2 + 5}{n+3}}$

5. (8+8 points) Calculate the limit of the following sequences:

a) $a_n = \left(\frac{3n+4}{3n+7}\right)^{2n}$ b) $b_n = \left(\frac{n^2+2n}{n^2+2}\right)^{n^2}$

6. (8+8+8 points) Let $a_1 = 2$ and $a_{n+1} = \sqrt{5a_n - 4}$

a) Prove that for the terms of this recursive sequence $1 < a_n < 4$ for all n.

b) Prove that the sequence is monotonically increasing.

c) Prove that the sequence has a limit and calculate it.

Calculus 1, Midterm test 1, B

20th October, 2025

1. (8+12 points)

a) Let $z = 1 - 3\,i$. Calculate $\dfrac{z+7}{z+2\,\overline{z}}$ and give the result in algebraic form.

b) $\sqrt[3]{-27\,i}$ = ? Give the roots in trigonometric form and also plot them.

2. (16 points) Let $a_n = \dfrac{16\,n^4 - 2\,n + 3}{8\,n^4 + 3\,n^3 + 7}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.001$.

3. (12 points) Find the limit of the following sequence: $a_n = n^3\left(\sqrt{n^8 + 9\,n} - \sqrt{n^8 - 5\,n}\right)$.

4. (14 points) Find the limit of the following sequence: $a_n = \sqrt[n]{\dfrac{5^n + n^3 + 2}{2\,n + 7}}$

5. (8+8 points) Calculate the limit of the following sequences:

a) $a_n = \left(\dfrac{2\,n+5}{2\,n+9}\right)^{3\,n}$ b) $b_n = \left(\dfrac{n^2+n}{n^2+3}\right)^{n^2}$

6. (8+8+8 points) Let $a_1 = 3$ and $a_{n+1} = \sqrt{8\,a_n - 12}$

a) Prove that for the terms of this recursive sequence $2 < a_n < 6$ for all n.

b) Prove that the sequence is monotonically increasing.

c) Prove that the sequence has a limit and calculate it.

Solutions

A 1. (8+12 points)

a) Let $z = 1 + 2\,i$. Calculate $\dfrac{z+6}{3\,z+\overline{z}}$ and give the result in algebraic form.

b) $\sqrt[3]{8\,i}\ = ?$ Give the roots in trigonometric form and also plot them.

Solution.

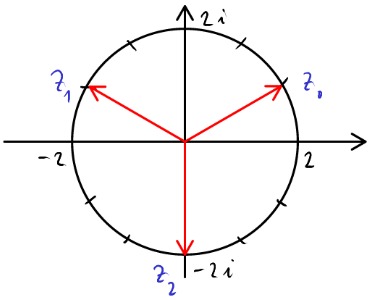
a) $\dfrac{z+6}{3\,z+\overline{z}} = \dfrac{1+2\,i+6}{3\,(1+2\,i)+(1-2\,i)} = \dfrac{7+2\,i}{4+4\,i} = \dfrac{1}{4}\cdot\dfrac{7+2\,i}{1+i}\cdot\dfrac{1-i}{1-i} = \dfrac{1}{4}\cdot\dfrac{9-5\,i}{1+1} = \dfrac{9}{8} - \dfrac{5\,i}{8}$

b) $8\,i = 8\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$

$z_k = \sqrt[3]{8\,i}\ = 2\left(\cos\dfrac{\frac{\pi}{2}+k\cdot 2\,\pi}{3} + i\sin\dfrac{\frac{\pi}{2}+k\cdot 2\,\pi}{3}\right),\ k = 0, 1, 2$ **(4 points)**

$\Longrightarrow z_0 = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right),\quad z_1 = 2\left(\cos\dfrac{5\,\pi}{6} + i\sin\dfrac{5\,\pi}{6}\right)$

$z_2 = 2\left(\cos\dfrac{9\,\pi}{6} + i\sin\dfrac{9\,\pi}{6}\right) = 2\left(\cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right)\right)$ **(4 points)**



(4 points)

B 1. (8+12 points)

a) Let $z = 1 - 3\,i$. Calculate $\dfrac{z+7}{z+2\,\overline{z}}$ and give the result in algebraic form.

b) $\sqrt[3]{-27\,i}\ = ?$ Give the roots in trigonometric form and also plot them.

Solution.

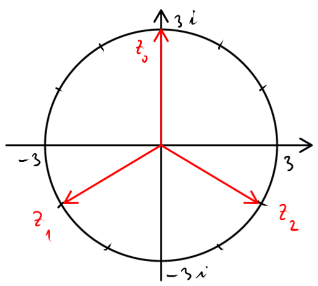
a) $\dfrac{z+7}{z+2\,\overline{z}} = \dfrac{1-3\,i+7}{1-3\,i+2\,(1+3\,i)} = \dfrac{8-3\,i}{3+3\,i} = \dfrac{1}{3}\cdot\dfrac{8-3\,i}{1+i}\cdot\dfrac{1-i}{1-i} = \dfrac{1}{3}\cdot\dfrac{5-11\,i}{1+1} = \dfrac{5}{6} - \dfrac{11\,i}{6}$

b) $-27\,i = 27\left(\cos\dfrac{3\,\pi}{2} + i\sin\dfrac{3\,\pi}{2}\right)$

$z_k = \sqrt[3]{-27\,i}\ = 3\left(\cos\dfrac{\frac{3\,\pi}{2}+k\cdot 2\,\pi}{3} + i\sin\dfrac{\frac{3\,\pi}{2}+k\cdot 2\,\pi}{3}\right),\ k = 0, 1, 2$ **(4 points)**

$\Longrightarrow z_0 = 3\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right),\quad z_1 = 3\left(\cos\dfrac{7\,\pi}{6} + i\sin\dfrac{7\,\pi}{6}\right)$

$z_2 = 3\left(\cos\dfrac{11\,\pi}{6} + i\sin\dfrac{11\,\pi}{6}\right)$ **(4 points)**



(4 points)

A 2. (16 points) Let $a_n = \dfrac{6n^3 - n + 10}{2n^3 + 5n^2 + 8}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.01$.

Solution. $a_n = \dfrac{6n^3 - n + 10}{2n^3 + 5n^2 + 8} = \dfrac{6 - \frac{1}{n^2} + \frac{10}{n^3}}{2 + \frac{5}{n} + \frac{8}{n^3}} \longrightarrow \dfrac{6 - 0 + 0}{2 + 0 + 0} = 3$ **(3 points)**

Let $\varepsilon > 0$. We have to find $N(\varepsilon) \in \mathbb{N}$ such that if $n > N$ then $|a_n - A| < \varepsilon$. $(A = 3)$ **(3 points)**

$$|a_n - A| = \left|\frac{6n^3 - n + 10}{2n^3 + 5n^2 + 8} - 3\right| = \left|\frac{6n^3 - n + 10 - 3\cdot(2n^3 + 5n^2 + 8)}{2n^3 + 5n^2 + 8}\right| = \left|\frac{-15n^2 - n - 14}{2n^3 + 5n^2 + 8}\right| =$$
$$= \frac{15n^2 + n + 14}{2n^3 + 5n^2 + 8}$$ **(4 points)**

$$\frac{15n^2 + n + 14}{2n^3 + 5n^2 + 8} \le \frac{15n^2 + n^2 + 14n^2}{2n^3 + 0 + 0} = \frac{30n^2}{2n^3} = \frac{15}{n} < \varepsilon \iff n > \frac{15}{\varepsilon}$$ **(4 points)**

so with the choice $N(\varepsilon) \ge \left[\dfrac{15}{\varepsilon}\right]$ the definition holds.

If $\varepsilon = 0.01$ then $N \ge \left[\dfrac{15}{0.01}\right] = 1500$. **(2 points)**

B 2. (16 points) Let $a_n = \dfrac{16n^4 - 2n + 3}{8n^4 + 3n^3 + 7}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.001$.

Solution. $a_n = \dfrac{16n^4 - 2n + 3}{8n^4 + 3n^3 + 7} = \dfrac{16 - \frac{1}{8n^3} + \frac{3}{16n^4}}{8 + \frac{3}{n} + \frac{7}{n^4}} \longrightarrow \dfrac{16 - 0 + 0}{8 + 0 + 0} = 2$ **(3 points)**

Let $\varepsilon > 0$. We have to find $N(\varepsilon) \in \mathbb{N}$ such that if $n > N$ then $|a_n - A| < \varepsilon$. $(A = 2)$ **(3 points)**

$$|a_n - A| = \left|\frac{16n^4 - 2n + 3}{8n^4 + 3n^3 + 7} - 2\right| = \left|\frac{16n^4 - 2n + 3 - 2\cdot(8n^4 + 3n^3 + 7)}{8n^4 + 3n^3 + 7}\right| = \left|\frac{-6n^3 - 2n - 11}{8n^4 + 3n^3 + 7}\right| =$$
$$= \frac{6n^3 + 2n + 11}{8n^4 + 3n^3 + 7}$$ **(4 points)**

$$\frac{6n^3 + 2n + 11}{8n^4 + 3n^3 + 7} \le \frac{6n^3 + 2n^3 + 11n^3}{8n^4 + 0 + 0} = \frac{19n^3}{8n^4} = \frac{19}{8n} < \varepsilon \iff n > \frac{19}{8\varepsilon}$$ **(4 points)**

so with the choice $N(\varepsilon) \geq \left[\frac{19}{8\,\varepsilon}\right]$ the definition holds.

If $\varepsilon = 0.001$ then $N \geq \left[\frac{19}{0.008}\right]$ **(2 points)** ≈ 2375

A 3. (12 points) Find the limit of the following sequence: $a_n = n^2\left(\sqrt{n^6 - 7\,n} - \sqrt{n^6 + 3\,n}\right)$.

Solution. $a_n = n^2\left(\sqrt{n^6 - 7\,n} - \sqrt{n^6 + 3\,n}\right) \cdot \frac{\sqrt{n^6 - 7\,n} + \sqrt{n^6 + 3\,n}}{\sqrt{n^6 - 7\,n} + \sqrt{n^6 + 3\,n}} =$

$$= n^2 \cdot \frac{\left(n^6 - 7\,n\right) - \left(n^6 + 3\,n\right)}{\sqrt{n^6 - 7\,n} + \sqrt{n^6 + 3\,n}} = n^2 \cdot \frac{-10\,n}{\sqrt{n^6 - 7\,n} + \sqrt{n^6 + 3\,n}} =$$

$$= \frac{n^3}{n^3} \cdot \frac{-10}{\sqrt{1 - \frac{7}{n^5}} + \sqrt{1 + \frac{3}{n^5}}} \longrightarrow \frac{-10}{\sqrt{1 - 0} + \sqrt{1 + 0}} = -5$$

B 3. (12 points) Find the limit of the following sequence: $a_n = n^3\left(\sqrt{n^8 + 9\,n} - \sqrt{n^8 - 5\,n}\right)$.

Solution. $a_n = n^3\left(\sqrt{n^8 + 9\,n} - \sqrt{n^8 - 5\,n}\right) \cdot \frac{\sqrt{n^8 + 9\,n} + \sqrt{n^8 - 5\,n}}{\sqrt{n^8 + 9\,n} + \sqrt{n^8 - 5\,n}} =$

$$= n^3 \cdot \frac{\left(n^8 + 9\,n\right) - \left(n^8 - 5\,n\right)}{\sqrt{n^8 + 9\,n} + \sqrt{n^8 - 5\,n}} = n^3 \cdot \frac{14\,n}{\sqrt{n^8 + 9\,n} + \sqrt{n^8 - 5\,n}} =$$

$$= \frac{n^4}{n^4} \cdot \frac{14}{\sqrt{1 + \frac{9}{n^7}} + \sqrt{1 - \frac{5}{n^7}}} \longrightarrow \frac{14}{\sqrt{1 + 0} + \sqrt{1 - 0}} = 7$$

A 4. (14 points) Find the limit of the following sequence: $a_n = \sqrt[n]{\frac{2^n + n^2 + 5}{n + 3}}$

Solution. Upper estimation, if n is large enough:

$$a_n = \sqrt[n]{\frac{2^n + n^2 + 5}{n + 3}} \leq \sqrt[n]{\frac{2^n + 2^n + 5 \cdot 2^n}{n}} = \sqrt[n]{\frac{7 \cdot 2^n}{n}} = \frac{\sqrt[n]{7} \cdot \sqrt[n]{2^n}}{\sqrt[n]{n}} = \frac{\sqrt[n]{7} \cdot 2}{\sqrt[n]{n}} \longrightarrow \frac{1 \cdot 2}{1} = 2 \text{ (7 points)}$$

Lower estimation:

$$a_n = \sqrt[n]{\frac{2^n + n^2 + 5}{n + 3}} \geq \sqrt[n]{\frac{2^n + 0 + 0}{n + 3\,n}} = \sqrt[n]{\frac{2^n}{4\,n}} = \frac{\sqrt[n]{2^n}}{\sqrt[n]{4} \cdot \sqrt[n]{4}} = \frac{2}{\sqrt[n]{4} \cdot \sqrt[n]{4}} \longrightarrow \frac{2}{1 \cdot 1} = 2 \text{ (6 points)}$$

So by the Sandwich Theorem, $a_n \longrightarrow 2$. **(1 point)**

B 4. (16 points) Find the limit of the following sequence: $a_n = \sqrt[n]{\frac{5^n + n^3 + 2}{2\,n + 7}}$

Solution. Upper estimation, if n is large enough:

$$a_n = \sqrt[n]{\frac{5^n + n^3 + 2}{2\,n + 7}} \leq \sqrt[n]{\frac{5^n + 5^n + 2 \cdot 5^n}{2\,n + 0}} = \sqrt[n]{\frac{4 \cdot 5^n}{2\,n}} = \frac{\sqrt[n]{4} \cdot \sqrt[n]{5^n}}{\sqrt[n]{2\,n}} = \frac{\sqrt[n]{4} \cdot 5}{\sqrt[n]{2} \cdot \sqrt[n]{n}} \longrightarrow \frac{1 \cdot 5}{1 \cdot 1} = 5$$

Lower estimation:

$$a_n = \sqrt[n]{\frac{5^n + n^3 + 2}{2n+7}} \geq \sqrt[n]{\frac{5^n + 0 + 0}{2n + 7n}} = \sqrt[n]{\frac{5^n}{9n}} = \frac{\sqrt[n]{5^n}}{\sqrt[n]{9} \cdot \sqrt[n]{n}} = \frac{5}{\sqrt[n]{9} \cdot \sqrt[n]{n}} \longrightarrow \frac{5}{1 \cdot 1} = 5$$

So by the Sandwich Theorem, $a_n \longrightarrow 5$.

A 5. (8+8 points) Calculate the limit of the following sequences:

a) $a_n = \left(\frac{3n+4}{3n+7}\right)^{2n}$ b) $b_n = \left(\frac{n^2+2n}{n^2+2}\right)^{n^2}$

Solution.

a) $a_n = \left(\frac{1+\frac{4}{3n}}{1+\frac{7}{3n}}\right)^{2n} = \frac{\left(\left(1+\frac{4}{3n}\right)^n\right)^2}{\left(\left(1+\frac{7}{3n}\right)^n\right)^2} \longrightarrow \frac{\left(e^{\frac{4}{3}}\right)^2}{\left(e^{\frac{7}{3}}\right)^2} = e^{\frac{8}{3}-\frac{14}{3}} = e^{-2}$

b) $b_n = \frac{\left(1+\frac{2}{n}\right)^{n^2}}{\left(1+\frac{2}{n^2}\right)^{n^2}} \geq \frac{\left(\left(1+\frac{2}{n}\right)^n\right)^n}{e^2} \geq \frac{3^n}{e^2} \longrightarrow \infty$, therefore $b_n \longrightarrow \infty$.

We use that since $\left(1+\frac{2}{n}\right)^n \longrightarrow e^2$, then the terms are greater then 3, if n is large enough.

B 5. (8+8 points) Calculate the limit of the following sequences:

a) $a_n = \left(\frac{2n+5}{2n+9}\right)^{3n}$ b) $b_n = \left(\frac{n^2+n}{n^2+3}\right)^{n^2}$

Solution.

a) $a_n = \left(\frac{1+\frac{5}{2n}}{1+\frac{9}{2n}}\right)^{2n} = \frac{\left(\left(1+\frac{5}{2n}\right)^n\right)^3}{\left(\left(1+\frac{9}{2n}\right)^n\right)^3} \longrightarrow \frac{\left(e^{\frac{5}{2}}\right)^3}{\left(e^{\frac{9}{2}}\right)^3} = e^{\frac{15}{2}-\frac{27}{2}} = e^{-6}$

b) $b_n = \frac{\left(1+\frac{1}{n}\right)^{n^2}}{\left(1+\frac{3}{n^2}\right)^{n^2}} \geq \frac{\left(\left(1+\frac{1}{n}\right)^n\right)^n}{e^3} \geq \frac{2^n}{e^3} \longrightarrow \infty$, therefore $b_n \longrightarrow \infty$.

We use that since $\left(1+\frac{1}{n}\right)^n \longrightarrow e$, then the terms are greater then 2, if n is large enough.

A 6. (8+8+8 points) Let $a_1 = 2$ and $a_{n+1} = \sqrt{5a_n - 4}$

a) Prove that for the terms of this recursive sequence $1 < a_n < 4$ for all n.

b) Prove that the sequence is monotonically increasing.

c) Prove that the sequence has a limit and calculate it.

Solution.

a) Boundedness: we prove by induction that $1 < a_n < 4$ for all n.

(1) $1 < a_1 = 2 < 4$

(2) Assume that $1 < a_n < 4$.

(3) Then $5 \cdot 1 - 4 < 5a_n - 4 < 5 \cdot 4 - 4 \implies 1 < 5a_n - 4 < 16 \implies 1 < a_{n+1} = \sqrt{5a_n - 4} < 4$

So (a_n) is bounded. **(8 points)**

b) Monotonicity: we prove by induction that (a_n) is monotonically increasing, that is, $a_n < a_{n+1}$ for all n.

(1) $a_1 = 2 < a_2 = \sqrt{5 \cdot 2 - 4} = \sqrt{6}$
(2) Assume that $a_n < a_{n+1}$
(3) Then $5\,a_n - 4 < 5\,a_{n+1} - 4 \implies a_{n+1} = \sqrt{5\,a_n - 4} < \sqrt{5\,a_{n+1} - 4} = a_{n+2} \implies a_{n+1} < a_{n+2}$
So (a_n) is monotonically increasing. **(8 points)**

c) Since (a_n) is monotonically increasing and bounded above then it is convergent.
Let $\lim\limits_{n\to\infty} a_n = A$. Then $A = \sqrt{5\,A - 4} \iff A^2 - 5\,A + 4 = (A-1)(A-4) = 0 \iff A_1 = 1, A_2 = 4$.
Since $a_1 = 2$ and the sequence is monotonically increasing then $A = 1$ cannot be the limit.
So $\lim\limits_{n\to\infty} a_n = 4$. **(8 points)**

B 6. (8+8+8 points) Let $a_1 = 3$ and $a_{n+1} = \sqrt{8\,a_n - 12}$
a) Prove that for the terms of this recursive sequence $2 < a_n < 6$ for all n.
b) Prove that the sequence is monotonically increasing.
c) Prove that the sequence has a limit and calculate it.

Solution.
a) Boundedness: we prove by induction that $2 < a_n < 6$ for all n.
(1) $2 < a_1 = 3 < 6$
(2) Assume that $2 < a_n < 6$.
(3) Then $8 \cdot 2 - 12 < 8\,a_n - 12 < 8 \cdot 6 - 12 \implies 4 < 8\,a_n - 12 < 36 \implies 2 < a_{n+1} = \sqrt{8\,a_n - 12} < 6$
So (a_n) is bounded. **(8 points)**

b) Monotonicity: we prove by induction that (a_n) is monotonically increasing, that is, $a_n < a_{n+1}$ for all n.
(1) $a_1 = 3 < a_2 = \sqrt{8 \cdot 3 - 12} = \sqrt{12}$
(2) Assume that $a_n < a_{n+1}$
(3) Then $8\,a_n - 12 < 8\,a_{n+1} - 12 \implies a_{n+1} = \sqrt{8\,a_n - 12} < \sqrt{8\,a_{n+1} - 12} = a_{n+2} \implies a_{n+1} < a_{n+2}$
So (a_n) is monotonically increasing. **(8 points)**

c) Since (a_n) is monotonically increasing and bounded above then it is convergent.
Let $\lim\limits_{n\to\infty} a_n = A$. Then $A = \sqrt{8\,A - 12} \iff A^2 - 8\,A + 12 = (A-2)(A-6) = 0 \iff A_1 = 2, A_2 = 6$.
Since $a_1 = 3$ and the sequence is monotonically increasing then $A = 3$ cannot be the limit.
So $\lim\limits_{n\to\infty} a_n = 6$. **(8 points)**