Calculus 1, Midterm test 2

24th November, 2025

1. (18 points) Where and what type of discontinuity does the following function have?

$$f(x) = \begin{cases} \dfrac{\sin(x-2)}{x^2+2x-8}, & \text{if } x \geq 0 \\ \dfrac{4\cosh(x^2)}{x+3}, & \text{if } x < 0 \end{cases}$$

2. (10 points) Calculate the derivative of the following function where it exists:

$$g(x) = \begin{cases} \sin(4x^2)\cos\dfrac{1}{3x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

(At $x = 0$ use the definition.)

3. (10+10+10 points) Calculate the following limits:

a) $\displaystyle\lim_{x\to 0} \frac{e^{3x} - 1 - \sin 3x}{x\ln(2x+1)}$ b) $\displaystyle\lim_{x\to 0} (\cosh x)^{\frac{2}{\sinh^2 x}}$ c) $\displaystyle\lim_{x\to\infty} \frac{\sinh(3x+1)}{\cosh(3x-5)}$

4. (12 points) Find the equation of the tangent line to the graph of the following function at $x_0 = 0$:

$$f(x) = \ln\left(\frac{x+2}{x^2+1}\right)\cdot\sqrt{x^2+1}$$

5. (15 points) Find the local extrema of the function $f(x) = \left(x^3 + 3x^2 + 3x - 3\right)e^x$.
Determine the intervals where the function increases or decreases.

6. (15 points) Find the inflection points of the function $f(x) = (x+2)\arctan(x)$.
Determine the intervals where the function is convex or concave.

Solutions

1. (18 points) Where and what type of discontinuity does the following function have?

$$f(x) = \begin{cases} \dfrac{\sin(x-2)}{x^2+2x-8}, & \text{if } x \geq 0 \\ \dfrac{4\cosh(x^2)}{x+3}, & \text{if } x < 0 \end{cases}$$

Solution. f is a composition of continuous functions, so it can have discontinuities at the zeros of the denominators (at $x = 2$ and $x = -3$) and at $x = 0$.
At $x = -4$ there is no discontinuity because the upper fraction is not defined at this point. **(3p)**

$$\lim_{x\to 2\pm} f(x) = \lim_{x\to 2\pm} \frac{\sin(x-2)}{(x-2)} \cdot \frac{1}{x+4} = 1 \cdot \frac{1}{6}$$ **(3p)**

$\Longrightarrow$ f has a removable discontinuity at $x = 2$ **(2p)**

$$\lim_{x\to -3\pm} f(x) = \lim_{x\to -3\pm} \frac{4\,\mathrm{ch}(x^2)}{x+3} = \pm\infty$$ **(3p)**

$\Longrightarrow$ f has an essential discontinuity at $x = -3$ **(2p)**

$$\lim_{x\to 0+} f(x) = f(0) = \frac{\sin 2}{8} \neq \frac{4}{3} = \frac{4\,\mathrm{ch}(0^2)}{0+3} = \lim_{x\to 0-} \frac{4\,\mathrm{ch}(x^2)}{x+3}$$ **(3p)**

$\Longrightarrow$ f has a jump discontinuity at $x = 0$ **(2p)**

2. (10 points) Calculate the derivative of the following function where it exists:

$$g(x) = \begin{cases} \sin(4x^2)\cos\dfrac{1}{3x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

(At $x = 0$ use the definition.)

Solution. If $x \neq 0$ then

$$g'(x) = 8x\cos(4x^2)\cos\frac{1}{3x} + \frac{1}{3x^2}\sin(4x^2)\sin\frac{1}{3x}$$ **(5p)**

If $x = 0$ then

$$g'(0) = \lim_{h\to 0} \frac{\sin(4h^2)\cos\frac{1}{3h}}{h} = \lim_{h\to 0} \frac{\sin(4h^2)}{4h^2} \cdot 4h\cos\frac{1}{3h} = \lim_{h\to 0} 4h\cos\frac{1}{3h} = 0$$ **(5p)**

3. (10+10+10 points) Calculate the following limits:

a) $\displaystyle\lim_{x\to 0} \frac{e^{3x} - 1 - \sin 3x}{x\ln(2x+1)}$ b) $\displaystyle\lim_{x\to 0} (\cosh x)^{\frac{2}{\sinh^2 x}}$ c) $\displaystyle\lim_{x\to\infty} \frac{\sinh(3x+1)}{\cosh(3x-5)}$

Solution.

a) $\lim\limits_{x\to 0} \dfrac{e^{3x}-1-\sin 3x}{x\ln(2x+1)} \overset{\text{"}\frac{0}{0}\text{"}, L'H}{=} \lim\limits_{x\to 0} \dfrac{3e^{3x}-3\cos 3x}{\ln(2x+1)+\frac{2x}{2x+2}} \overset{\text{"}\frac{0}{0}\text{"}, L'H}{=}$ **(5p)**

$= \lim\limits_{x\to 0} \dfrac{9e^{3x}+9\sin 3x}{\frac{2}{2x+2}+\frac{2(2x+2)-2x\cdot 2}{(2x+2)^2}} = \dfrac{9+0}{1+\frac{4-0}{4}} = \dfrac{9}{2}$ **(5p)**

b) The limit has the form 1^∞: $(\cosh x)^{\frac{2}{\sinh^2 x}} = e^{\ln\left((\cosh x)^{\frac{2}{\sinh^2 x}}\right)} = e^{\left(\frac{2}{\sinh^2 x}\ln(\cosh x)\right)}$ **(3p)**

The limit of the power has the form $\frac{0}{0}$:

$\lim\limits_{x\to 0} \dfrac{2\ln(\cosh x)}{\sinh^2 x} \overset{L'H}{=} \lim\limits_{x\to 0} \dfrac{\frac{2}{\cosh x}\cdot\sinh x}{2\sinh x\cosh x} = \lim\limits_{x\to 0}\dfrac{1}{\cosh^2 x} = 1$ **(5p)**

$\Longrightarrow \lim\limits_{x\to 0} (\cosh x)^{\frac{2}{\sinh^2 x}} = e^1 = e$ **(2p)**

c)

$\lim\limits_{x\to\infty} \dfrac{\sinh(3x+1)}{\cosh(3x-5)} = \lim\limits_{x\to\infty} \dfrac{e^{3x+1}-e^{-(3x+1)}}{e^{3x-5}+e^{-(3x-5)}}$ **(3p)** $= \lim\limits_{x\to\infty} \dfrac{e^{3x}}{e^{3x}}\,\dfrac{e-e^{-6x-3}}{e^{-5}+e^{-6x+5}}$ **(4p)** $= \dfrac{e-0}{e^{-5}+0} = e^6$ **(3p)**

4. (12 points) Find the equation of the tangent line to the graph of the following function at $x_0 = 0$:

$$f(x) = \ln\left(\frac{x+2}{x^2+1}\right)\cdot\sqrt{x^2+1}$$

Solution. $f'(x) = \dfrac{x^2+1}{x+2}\cdot\dfrac{(x^2+1)-(x+2)\cdot 2x}{(x^2+1)^2}\cdot\sqrt{x^2+1} + \ln\left(\dfrac{x+2}{x^2+1}\right)\cdot\dfrac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x$ **(6p)**

$f(0) = \ln(2)$, $f'(0) = \dfrac{1}{2}$ **(1p)**

The equation of the tangent line is $y = f(0) + f'(0)(x-0)$, that is, $y = \ln 2 + \dfrac{1}{2}x$ **(3p)**

5. (15 points) Find the local extrema of the function $f(x) = \left(x^3+3x^2+3x-3\right)e^x$.
Determine the intervals where the function increases or decreases.

Solution. $f'(x) = e^x x(x+3)^2 = 0 \iff x_1 = 0$ and $x_2 = -3$

x	x<-3	x=-3	-3<x<0	x=0	x>0
f'	–	0	–	0	+
f	↘		↘	min:0	↗

f is strictly monotonically decreasing on $(-\infty, 0]$ and strictly monotonically increasing on $[0, \infty)$.

6. (15 points) Find the inflection points of the function $f(x) = (x+2)\arctan(x)$.
Determine the intervals where the function is convex or concave.

Solution.

$$f'(x) = \arctan(x) + \frac{x+2}{1+x^2}$$

$$f''(x) = \frac{1}{1+x^2} + \frac{1\cdot(1+x^2)-(x+2)\cdot 2x}{(1+x^2)^2}$$

$$= \frac{(1+x^2)+(1+x^2)-2x^2-4x}{(1+x^2)^2} = \frac{2+2x^2-2x^2-4x}{(1+x^2)^2} =$$

$$= \frac{2-4x}{(1+x^2)^2} = 0 \iff x = \frac{1}{2}$$

x	$x<\frac{1}{2}$	$x=\frac{1}{2}$	$x>\frac{1}{2}$
f''	+	0	−
f	$\cup$	infl.	$\cap$

f is convex on $\left(-\infty, \frac{1}{2}\right)$ and concave on $\left(\frac{1}{2}, \infty\right) \Longrightarrow f$ has an inflection point at $\frac{1}{2}$.