Differentiation - practice exercises

1. Analyze the following function and plot its graph: $f(x) = (x^2 - x)\, e^x$.

Solution.

$D_f = \mathbb{R};\ \ f(x) = 0 \iff x = 0 \text{ or } x = 1;\ \ \lim_{x\to+\infty} f(x) = \infty,\ \ \lim_{x\to-\infty} f(x) = 0$

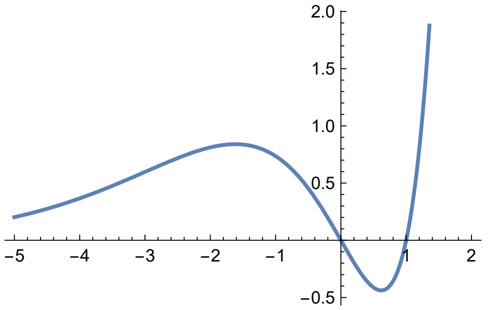
$f'(x) = e^x (x^2 + x - 1) = 0 \iff x = x_1 = \frac{1}{2}\left(-1 - \sqrt{5}\right) \text{ or } x = x_2 = \frac{1}{2}\left(-1 + \sqrt{5}\right)$

x	$x<x_1$	$x=x_1$	$x_1<x<x_2$	$x=x_2$	$x>x_2$
f'	+	0	−	0	+
f	↗	loc. max	↘	loc. min	↗

$(x_1 \approx -1.61803,\ \ x_2 \approx 0.618034,\ \ f(x_1) \approx 0.839962,\ \ f(x_2) \approx -0.437971)$

$f''(x) = e^x \cdot x\,(x + 3) = 0 \iff x_1 = -3,\ x_2 = 0$

x	$x<-3$	$x=-3$	$-3<x<0$	$x=0$	$0<x$
f''	+	0	−	0	+
f	∪	infl:0	∩	infl: $\frac{12}{e^3}$	∪



2. Analyze the following function and sketch its graph: $f(x) = x\, e^{-2x^2}$.

Solution.

$D_f = \mathbb{R};\ \ f(x) = 0 \iff x = 0;\ \ f$ is odd

$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x}{e^{2x^2}} \overset{L'H}{=} \lim_{x\to\pm\infty} \frac{1}{e^{2x^2}\cdot 4x} = 0$

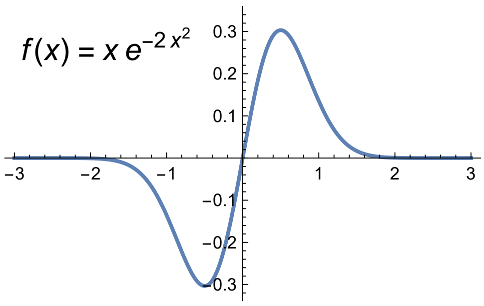
$f'(x) = e^{-2x^2}(1 - 4x^2) = 0 \iff x = \pm\frac{1}{2}$

x	$x<-\frac{1}{2}$	$x=-\frac{1}{2}$	$-\frac{1}{2}<x<\frac{1}{2}$	$x=\frac{1}{2}$	$x>\frac{1}{2}$
f'	−	0	+	0	−
f	↘	min: $-\frac{1}{2\sqrt{e}} \approx -0.303$	↗	max: $\frac{1}{2\sqrt{e}} \approx 0.303$	↘

$f''(x) = 4\,e^{-2x^2}\,x\left(-3+4x^2\right) = 0 \iff x = 0 \text{ or } x = \pm\frac{\sqrt{3}}{2}$

x	$x<-\frac{\sqrt{3}}{2}$	$x=-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}<x<0$	$x=0$	$0<x<\frac{\sqrt{3}}{2}$	$x=\frac{\sqrt{3}}{2}$	$x>\frac{\sqrt{3}}{2}$
f''	–	0	+	0	–	0	+
f	∩	infl:≈-0.193	∪	infl:0	∩	infl:≈0.193	∪

The graph of f



$\Longrightarrow$ the range of f is $\left[-\frac{1}{2\sqrt{e}}, \frac{1}{2\sqrt{e}}\right]$

3. A rectangular box with a square base and no top needs to be made using 300 square centimeters of paper. Find the lengths of the edges if the volume of the rectangular box is maximal.

Solution. The surface area of the box with base x and height y is

$A = x^2 + 4\,x\,y = 300 \implies y = \frac{300 - x^2}{4\,x}$

The volume of the box is $V = x^2\,y$

We want to find the maximum of the function $V(x) = x^2 \cdot \frac{300 - x^2}{4\,x} = \frac{1}{4}\left(300\,x - x^3\right)$.

$V'(x) = \frac{1}{4}\left(300 - 3\,x^2\right) = 0 \implies x = 10$, since $x > 0$.

$V''(x) = \frac{1}{4}\cdot(-6\,x) \implies V''(10) = \frac{1}{4}\cdot(-60) < 0 \implies V$ has a local maximum at $x = 10$.

The side lengths of the box are: $x = 10$ cm, $y = 3$ cm.

4. The sum of the lengths of the edges of a square prism is 1m. What are the edge lengths of the prism with maximal volume? What is this maximal volume?

Solution. Let x denote the side of the square base and let y denote the height of the prism. Then the volume of the prism is $V = x^2\,y$ and the sum of the lengths of the edges is $8\,x + 4\,y = 1$.

From here $y = \frac{1 - 8\,x}{4}$. Substituting this into the volume, we get that

$V(x) = x^2 \cdot \frac{1 - 8\,x}{4} = \frac{1}{4}\left(x^2 - 8\,x^3\right)$.

We want to find the maximum of this function if $0 < x < \frac{1}{8}$.

$V'(x) = \frac{1}{4}(2x - 24x^2) = \frac{1}{4} \cdot 2x(1 - 12x) = 0 \iff x_1 = 0, \quad x_2 = \frac{1}{12}$

Because of the conditions, $x = 0$ cannot be the case.

$V''(x) = \frac{1}{4}(2 - 48x)$ and $V''\left(\frac{1}{12}\right) = \frac{1}{4}\left(2 - 48 \cdot \frac{1}{12}\right) = -\frac{1}{2} < 0$, so V has a maximum at $x = \frac{1}{12}$.

The sides of the prism with maximal volume are $\frac{1}{12}$, $\frac{1}{12}$ and $\frac{1}{12}$ m (so it is a cube) and the maximum of the

volume is $\frac{1}{12^3}$ m^3.

5. The radius of the base circle of a right circular cone is 2 meters,
and its height is 5 meters. Determine the dimensions of the cylinder with the maximum volume that can be inscribed in the cone.

Solution. The height of the cone is $m = 5$ and the radius of the base circle is $r = 2$.
Let x and y respectively denote the radius of the base circle and the height of the cylinder and let α denote the angle formed by the slant height of the cone with the plane of its base.

Then $\tan \alpha = \frac{m}{r} = \frac{y}{r - x}$.

The volume of the cylinder that we want to maximize is

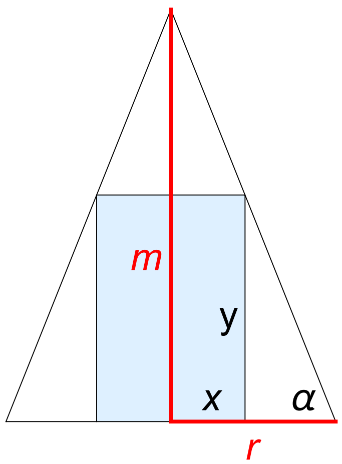
$V(x) = \pi x^2 y = \pi\left(\frac{5}{2} x^2 (2 - x)\right) = \pi\left(-\frac{5}{2} x^3 + 5x^2\right)$, where $0 < x < 2$.

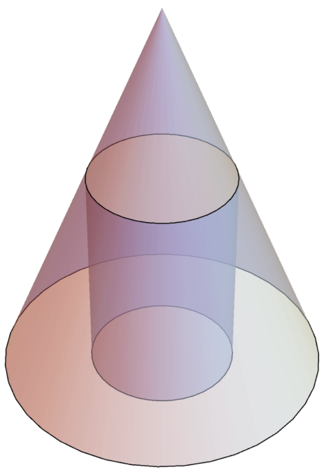
Then $V'(x) = \pi\left(-\frac{15}{2} x^2 + 10x\right) = 0$, from where $x = \frac{4}{3}$ (since $x > 0$).

At this point the function has a maximum, since

$V''(x) = \pi(-15x + 10)$, so $V''\left(\frac{4}{3}\right) = -10\pi < 0$.

The radius of the base circle and the height of the cylinder with maximum volume are
$x = \frac{4}{3}$ and $y = \frac{5}{3}$ meters, respectively.





6. You want to make a cylindrical tin cup with closed top of volume 1 liter.

What is the minimal possible surface area of the cup?

Solution. If r is the radius and h is the height of the cylinder then the volume is

$V = r^2 h \pi = 1 \implies h = \frac{1}{r^2 \pi}$, so the surface is

$s(r) = 2 r^2 \pi + 2 r \pi h = 2 r^2 \pi + \frac{2}{r}$ where $r \in (0, \infty)$.

$\implies s'(r) = 4 r \pi - \frac{2}{r^2} = 0$ if $r = \frac{1}{\sqrt[3]{2\pi}}$

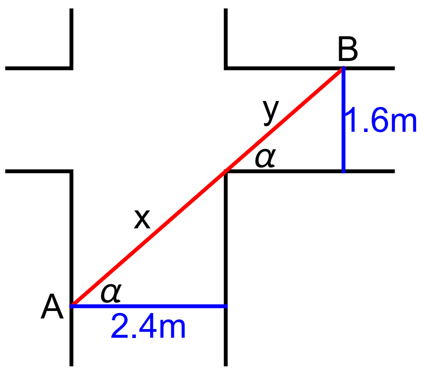
$s''(r) = 4\pi + \frac{4}{r^3} \implies s''\left(\frac{1}{\sqrt[3]{2\pi}}\right) > 0$

$\implies s$ has a local minimum at $r = \frac{1}{\sqrt[3]{2\pi}}$ and the minimal surface is $s(r) = 3 \cdot \sqrt[3]{2\pi}$.

This is a global minimum on $(0, \infty)$.

7.* The widths of two perpendicular corridors are 2.4 m and 1.6 m, respectively.
What is the longest ladder that can be moved (in a horizontal position) from one corridor to another?

Solution.



The ladder is denoted by the line segment AB in the figure. The lengths x and y can be expressed with α, so the length of the ladder is $AB = f(\alpha) = \frac{2.4}{\cos\alpha} + \frac{1.6}{\sin\alpha} \implies$

$f'(\alpha) = \frac{2.4 \sin\alpha}{\cos^2\alpha} - \frac{1.6\cos\alpha}{\sin^2\alpha} = \frac{0.8}{\sin^2\alpha \cos^2\alpha}(3\sin^3\alpha - 2\cos^3\alpha)$

$f'(\alpha) = 0$ if $3\sin^3\alpha = 2\cos^3\alpha \implies \tan\alpha = \sqrt[3]{\frac{2}{3}} \implies \alpha = \arctan\left(\sqrt[3]{\frac{2}{3}}\right) \approx 0.718025$

$\implies \alpha \approx 41°$. The length of the ladder is $AB \approx 3.2 + 2.4 = 5.6$ m

Remark. f has a local minimum at $\alpha \approx 0.718$. f' changes sign from negative to positive.

The graph of f:

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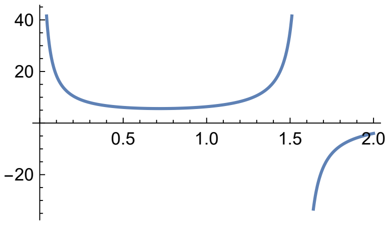
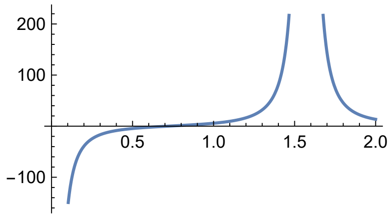

40
20
−20
0.5
1.0
1.5
2.0


200
100
−100
0.5
1.0
1.5
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