Practice exercises 1.

Logic and sets

1. Let A, B, C be some sets. Using set operations (intersection, union, complement, etc.) define the following sets:
a) The set of elements of B which are not included in either A or C.
b) The set of elements which belong to exactly two of the sets A, B, C.
c) The set of elements which are not included in all of the three sets.
d) Elements which belong to at most one of the sets.

2. Write down the following statements with logical formulas:
a) There exists an odd natural number larger than 10.
b) Every odd number, which is larger than one, is a prime number.
Write down also the negations of the above statements, both with words and with logical formulas.

3. Put the following statements into words:
a) $\forall x \in \mathbb{R}\left((x > 0) \Longrightarrow \left(\exists k \in \mathbb{N}\left(2^{-k} < x\right)\right)\right)$
b) $\exists k \in \mathbb{N}\left(\forall x \in \mathbb{R}\left((x > 0) \Longrightarrow \left(2^{-k} < x\right)\right)\right)$
Decide whether the statements are true or false. Write down also the negations of the above statements, both with words and with logical formulas.

4. Let $P(x)$ mean that x is an even number, and let $H(x)$ mean that x is divisible by 6. Put the following statements into words:
a) $P(4) \wedge H(12)$
b) $\forall x\,(P(x) \Longrightarrow H(x))$
c) $\exists x\,(P(x) \Longrightarrow H(x))$
d) $\exists x\,(H(x) \Longrightarrow \neg P(x))$
Decide whether the statements are true or false. Write down also the negations of the above statements, both with words and with logical formulas.

Proof by induction

5. Prove by induction that $1^2 + 3^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$.

6. Prove by induction that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

7. Prove by induction that for all $n > 1$ we have $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$.

8. Prove by induction that for all $n > 1$ we have $\frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1}$.

9. Let $a_0 = 1$ and $a_{n+1} = \sqrt{3a_n + 10}$. Prove that the sequence (a_n) is monotonically increasing.

Inequalities

10. We know that a, b, $c > 0$ and $a + b + c = 18$. Find the values of a, b and c such that the following expressions are maximal:

a) $a\,b\,c$ b) $a^2\,b\,c$ c) $\frac{a\,b\,c}{a\,b + b\,c + a\,c}$

11. We know that a, b, $c > 0$ and $a\,b\,c = 10$. ind the values of a, b and c such that the following expressions are minimal:

a) $a + b + c$ b) $2\,a + b + c$ c) $a^2 + b^2 + c^2$

12. a) Which rectangle has maximal area that we can write in a circle of radius 1?
b) Which rectangle has maximal perimeter that we can write in a circle of radius 1?