Practice exercises 2.

Supremum, infimum

1. Are the following sets bounded below or above? If so, determine their supremum and infimum. Do these sets have a minimum or a maximum?

   a) $H=\left\{\frac{1}{n}+(-1)^n:\ n\in\mathbb{N}\right\}\subset\mathbb{R}$ *b)* $H=\left\{\frac{(-1)^n}{n}+1: n\in\mathbb{N}\right\}\subset\mathbb{R}$

   c) $H=\left\{\frac{1}{2^n}+\frac{1}{2^m}: m, n\in\mathbb{N}\right\}\subset\mathbb{R}$ *d)* $H=\left\{\frac{x^2+1}{3x^2+2}: x\in\mathbb{R}\right\}\subset\mathbb{R}$

   e) $H=\left\{\frac{2x+3}{3x+1}: x\in\mathbb{Z}\right\}\subset\mathbb{R}$ *f)* $H=\left\{\frac{x}{y}: 0<x<1,\ 0<y<1\right\}\subset\mathbb{R}$

   g) $H=\left\{\frac{p}{q}: p, q\in\mathbb{Z},\ p^2<q^2\right\}\subset\mathbb{R}$ *h)* $H=\left\{r\in Q^+: r^2<2\right\}\subset\mathbb{R}$

Number sequences, part 1.

2. Find the limit of (a_n) and using the definition, provide a threshold index N for a given $\varepsilon>0$.

   a) $a_n=\frac{7n+4}{2n-1}$ b) $a_n=\frac{3n^2+4n+7}{n^2+n+1}$

   c) $a_n=\frac{n^2-10^8}{5n^6+2n^3-1}$ d) $a_n=1+(-1)^{n+1}\cdot 2^{3-n}$

3. Using the definition of the limit, show that

   a) $\lim\limits_{n\to\infty}\left(6n^3+3n\right)=\infty$ b) $\lim\limits_{n\to\infty}\sqrt{n^2-n}=\infty$ c) $\lim\limits_{n\to\infty}\left(n^3-3n^2+5n+9\right)=\infty$

   d) $\lim\limits_{n\to\infty}\frac{n^3+3n}{n^2+2}=\infty$ e) $\lim\limits_{n\to\infty}\frac{1+n^2-3n^3}{n^2+3n+7}=-\infty$

4. Formulate the following statements without negation:

   a) $\lim\limits_{n\to\infty}a_n\neq A\in\mathbb{R}$

   b) (a_n) is divergent

5. Calculate the limit of the following sequences:

   a) $a_n=\frac{n+3}{4n^2+7n+6}$ b) $a_n=\frac{n-5n^4}{n^4+8n^3+1}$ c) $a_n=\frac{1-n^3}{70-n^2+n}$

   d) $a_n=\frac{-n^7+n^6-3}{n^5-n^2+2}$ e) $a_n=\frac{\left(2n^3+3\right)^2}{(3n+6)^6}$ f) $a_n=\frac{(n+1)!}{(3-2n)\,n!}$

   g) $a_n=\frac{\binom{n}{2}}{\binom{n}{3}}$ h) $a_n=\frac{\binom{2n}{4}}{\binom{n+1}{2}\binom{n-1}{2}}$ i) $a_n=\sqrt[3]{\frac{2n^2+6}{3n^2+2n}}$

j) $a_n = \dfrac{n^{3/2} + n^2 + 1}{\sqrt{1+n^2} + 2\sqrt{n^3+2}}$ k) $a_n = \dfrac{\sqrt[4]{n^3+6}}{\sqrt[3]{n^2+3n+2}}$ l) $a_n = \dfrac{9\sqrt[3]{n} - 3\sqrt{2n} + 1}{\sqrt[4]{n} + \sqrt{3n}}$

6. Decide whether the following sequences converge and if so, find their limit:

a) $a_n = \sqrt{n^2+n+1} - \sqrt{n^2-n+1}$ b) $a_n = \sqrt{n^2-7n+1} - \sqrt{n^2-n+4}$

c) $a_n = \sqrt{2n^2+3n+1} - \sqrt{n^2+1}$ d) $a_n = (3n+1)\left(n - \sqrt{n^2+1}\right)$

e) $a_n = \dfrac{1}{n - \sqrt{n^2+3n+5}}$ f) $a_n = \sqrt[3]{n^3+3n^2-1} - \sqrt[3]{n^3-2n^2+3n+2}$

7. Prove that $\lim_{n\to\infty} \dfrac{a^n}{n!} = 0$, where $a \in \mathbb{R}$.

8. Calculate the limit of the following sequences:

a) $a_n = \dfrac{\sin(n)}{n}$ b) $a_n = \dfrac{n^2-5}{2n^3+6n}\cos(n^4+5n+8)$

c) $a_n = \dfrac{\log(n+1)}{n}$ d) $a_n = \dfrac{\log_{10}(n^2)+3}{\log_3(n)}$ e) $a_n = \dfrac{(-3)^{n+1} + 2^{2n+3}}{8+5^n}$

f) $a_n = \dfrac{3^{2n}+n^2+1}{3^n+9^n}$ g) $a_n = \dfrac{7^n+n^7+7}{4^n+3n^2+5}$ h) $a_n = \dfrac{4^{n-1}+n^5\cdot 3^{n+2}}{2^{2n+3}+2^{n-2}}$

i) $a_n = \dfrac{n^3 2^n + 3^n}{2^{2n}-3n^2}$ j) $a_n = \dfrac{2n! + n^{20}}{n^n}$ k) $a_n = \dfrac{(2^n+7^n)^2}{n!}$

9. True or false?

a) If $a_n \longrightarrow A$ then $a_n^2 \longrightarrow A^2$. b) If $a_n^2 \longrightarrow A^2$ then $a_n \longrightarrow A$.

c) If $a_n > 0$ and $b_n \longrightarrow \infty$ then $a_n b_n \longrightarrow \infty$. d) If $a_n \longrightarrow 0$ then $\dfrac{1}{a_n} \longrightarrow \infty$.

e) If $a_n \longrightarrow \infty$ then $\dfrac{1}{a_n} \longrightarrow 0$. f) If $a_n > 0$ and (a_n) is convergent then $\lim_{n\to\infty} a_n > 0$.

10. Is there a convergent sequence that can be written in the following form?

a) $a_n + b_n$ where both (a_n) and (b_n) are divergent

b) $a_n + b_n$ where (a_n) is convergent and (b_n) is divergent

c) $a_n \cdot b_n$ where both (a_n) and (b_n) are divergent

d) $a_n \cdot b_n$ where (a_n) is convergent and (b_n) is divergent

11. Prove the following statements:

a) If $a_n \longrightarrow A$ then $\sqrt[3]{a_n} \longrightarrow \sqrt[3]{A}$.

b) If $c_n \longrightarrow C > 0$ and $d_n \longrightarrow \infty$ then $c_n d_n \longrightarrow \infty$.