Practice exercises 3.

The limits $\sqrt[n]{p} \xrightarrow{n\to\infty} 1$ $(p>0)$ and $\sqrt[n]{n} \xrightarrow{n\to\infty} 1$

1. Calculate the following limits:

a) $a_n = \sqrt[2n]{2n}$ b) $a_n = \sqrt[n]{2n}$ c) $a_n = \sqrt[2n]{n}$

d) $a_n = \sqrt[5n]{3n}$ e) $a_n = \sqrt[n]{2n^3+3}$ f) $a_n = \sqrt[n]{\dfrac{2n^2+6}{3n^2+2n}}$

g) $a_n = \sqrt[n]{\dfrac{5n^2+4n-5}{n^3+6n^2-n}}$ h) $a_n = \sqrt[n^2]{n}$ i) $a_n = \sqrt[n]{n^3-n^2+4n+1}$

j) $a_n = \sqrt[n]{3^n+2^n}$ k) $a_n = \sqrt[n]{3^n-2^n}$ l) $a_n = \sqrt[n]{4^n+n^2+3n}$

m) $a_n = \sqrt[2n+1]{n^2+\cos n}$ n) $a_n = \sqrt[n]{1+\dfrac{1}{2}+\ldots+\dfrac{1}{n}}$

Theorems

2. a) Prove that if for all $n\in\mathbb{N}$ $a_n \ge 0$ and $\lim\limits_{n\to\infty} a_n = A$ then $\lim\limits_{n\to\infty}\sqrt[n]{a_n} = 1$ if $A>0$.

b) Give examples to show that the limit may change if $A=0$.

3. a) Prove that if $\lim\limits_{n\to\infty} a_n = A$ then $\lim\limits_{n\to\infty} a_n^n = \begin{cases} 0, & \text{if } |A|<1 \\ \infty, & \text{if } A>1 \\ \text{does not exist}, & \text{if } A\le -1 \end{cases}$.

b) Give examples to show that the limit may change or doesn't exist if $A=1$ or $A=-1$.

Recursive sequences

6. Investigate the convergence of the following sequences and calculate the limit if it exists.

a) $a_1 = 6,\ \ a_{n+1} = 5 - \dfrac{6}{a_n},\ \ n = 1, 2, \ldots$

b) $a_1 = \dfrac{4}{3},\ \ a_{n+1} = \dfrac{3+a_n^2}{4},\ \ n = 1, 2, \ldots$

c) $a_1 = 1,\ \ a_{n+1} = \sqrt{6+a_n},\ \ n = 1, 2, \ldots$

d) $a_1 = -3,\ \ a_{n+1} = \dfrac{5-6a_n^2}{13},\ \ n = 1, 2, \ldots$

e) $a_1 = 1,\ \ a_{n+1} = \dfrac{a_n}{1+a_n},\ \ n = 1, 2, \ldots$

f) $b_1 = 2,\ \ b_{n+1} = 4 + \sqrt{b_n - 2} - \dfrac{4}{\sqrt{n+4}}$

7.* Let $A>0$, $x_1 = 1$ and $x_{n+1} = \dfrac{x_n + \frac{A}{x_n}}{2}$. Prove that $x_n \longrightarrow \sqrt{A}$.

The limit $\left(1+\frac{x}{n}\right)^n \xrightarrow{n\to\infty} e^x$

4. Calculate the limits of the following sequences.

a) $a_n = \left(1+\dfrac{1}{6n^2}\right)^{6n^2+2}$, $b_n = \left(\dfrac{n+5}{n-4}\right)^{n+3}$, $c_n = \left(\dfrac{3n+5}{3n-4}\right)^{3n}$, $d_n = \left(\dfrac{3n-1}{3n+2}\right)^{2n}$

b) $a_n = \left(1+\dfrac{1}{n}\right)^{n^2}$, $b_n = \left(1+\dfrac{1}{n^2}\right)^{n}$, $c_n = \left(1-\dfrac{1}{n^4}\right)^{n^3}$, $d_n = \left(1-\dfrac{1}{n^4}\right)^{n^5}$

c) $a_n = \left(0.9+\dfrac{1}{n}\right)^{n}$, $b_n = \left(2-\dfrac{1}{n}\right)^{n}$, $c_n = \left(\dfrac{4n+1}{7n+5}\right)^{n}$, $d_n = \left(\dfrac{6n+1}{4n+5}\right)^{n}$

d) $a_n = \left(\dfrac{3n^2+1}{3n^2-2}\right)^{3n^2}$, $b_n = \left(\dfrac{3n^2+1}{3n^2-2}\right)^{3n^3}$, $c_n = \left(\dfrac{3n^2-2}{3n^2+1}\right)^{3n^3}$, $d_n = \left(\dfrac{3n^2-1}{3n^2+2}\right)^{3n}$

5.* Using the formula $\left(1+\dfrac{1}{z_n}\right)^{z_n} \longrightarrow e$ (for $z_n \longrightarrow \infty$) evaluate the following limits:

a) $a_n = \left(\dfrac{2^n+3}{2^n+1}\right)^{n}$ b) $a_n = \left(\dfrac{n^2-n+1}{n^2+n+1}\right)^{n}$ c) $a_n = \left(\dfrac{n^2+3n-4}{n^2-n+2}\right)^{4n+1}$ d) $a_n = \left(\dfrac{n^2+\sqrt{n}+1}{n^2-1}\right)^{4n^2+3}$