Practice exercises 4.

Limit superior and limit inferior

Find the limit inferior and limit superior of the following sequences.

a) $a_n = \dfrac{(-1)^n}{n} + \dfrac{1+(-1)^n}{2}$

b) $a_n = \left(3 + (-1)^n\right) n$

c) $a_n = 1 + 2\,(-1)^n + 3\,(-1)^{\frac{n(n+1)}{2}}$

d) $a_n = \cos\left(\dfrac{n\pi}{2}\right) \cdot \dfrac{2n^2-3}{n^2+n+8}, \quad b_n = \cos\left(\dfrac{n\pi}{2}\right) \cdot \dfrac{2n^2-3}{n^3+n+8}$

e) $a_n = \dfrac{(-3)^n+8}{5+4^n}, \quad b_n = \dfrac{(-4)^n+8}{5+4^n}$

f) $a_n = \sqrt{\dfrac{n^3+(-1)^n n^3}{3n^3+n+7}}$

Additional exercises

1.* Let (a_n) be a sequence of positive terms and let

$$A_n = \frac{a_1+a_2+\ldots+a_n}{n}, \quad G_n = \sqrt[n]{a_1\, a_2 \ldots a_n}, \quad H_n = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}.$$

a) Prove that if $\lim_{n\to\infty} a_n = A \in \mathbb{R}$ or $\lim_{n\to\infty} a_n = +\infty$ then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} A_n = \lim_{n\to\infty} G_n = \lim_{n\to\infty} H_n$.

b) Using this result prove that $\lim_{n\to\infty} \dfrac{1}{\sqrt[n]{n!}} = 0$ and $\lim_{n\to\infty} \dfrac{n}{\sqrt[n]{n!}} = e$.

2.* For all $r > 0$ show examples for sequences $a_n \longrightarrow 0+$ and $b_n \longrightarrow 0$ such that

$$a_n^{\,b_n} \longrightarrow r.$$

3.* For all $n \in \mathbb{N}$ we define the value of a_n by placing a decimal point in front of the index n written in the decimal number system and then a zero digit in front of it, and we interpret the number thus obtained in the decimal number system. For example $a_{4523} = 0.4523$ and $a_{100} = 0.100$. Find the accumulation points of the number sequence (a_n).

4.* Consider the following number sequence:

$$\frac{0}{1}, \quad \frac{0}{2}, \frac{1}{2}, \quad \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \quad \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \quad \ldots \quad \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \quad \frac{0}{n+1}, \frac{1}{n+1}, \ldots$$

a) Is 0 an accumulation point of the sequence?

b) Is 1 an accumulation point of the sequence?

c) Is the sequence convergent?

d) Exactly what real numbers are the accumulation points of the sequence?

Give reasons for your answers.

5.* Is there a number sequence whose real accumulation points are

a) the integers;

b) the rational numbers;

c) the points of the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$;

d) the points of the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} \cup \{0\}$;

e) the $[0, 1]$ closed interval?

If the answer is yes, then construct such a sequence.