Practice exercises 9.

The derivative

1. a) Let $f(x) = \sqrt[3]{x}$. Use the definition of the derivative to show that $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ if $x \neq 0$.

   Prove that $f'(0)$ does not exist.
   b) Prove that $(\cos x)' = -\sin x$.

2. Differentiate the following functions:

a) $f(x) = \dfrac{1}{x^3}$

b) $f(x) = (x^2 + 1)^{17}$

c) $f(x) = \sqrt{\dfrac{1}{x} \cdot \sqrt[3]{x}}$

d) $f(x) = \sqrt[4]{3x^2 + 5x}$

e) $f(x) = \tan x$

f) $f(x) = (x^3 + 3x)(\sin x + \cos x)$

g) $f(x) = \cos(x^3 + 3x - 1)$

h) $f(x) = \cos(2x)\sin(x^2 - 1)$

i) $f(x) = \sin^5(x^3)$

j) $f(x) = \tan(x^2 + 1)\sin\dfrac{1}{x}$

k) $f(x) = \cot\left(\dfrac{x^2 + 3}{\sqrt{\sin(2x - 1) + 7}}\right)$

l) $f(x) = \tan 3x \cos 5x \sin 7x$

3. Differentiate the following functions.

a) $\ln \sqrt{\cos x}$

b) $f(x) = \ln \dfrac{1 + \cos x}{1 - \sin x}$

c) $f(x) = \ln \sqrt[4]{\sin^3 x \cos^3 x}$

d) $f(x) = e^{3x^4 + x + 1}\ln(x^2 + 1)$ e) $f(x) = \dfrac{\sin(x)\ln(1 + \cos^2(x^3))}{x}$

f) $f(x) = \dfrac{\arctan(3x^2 + 4)\cos\left(\sqrt{2x + 3}\right)}{\log(\sin 3x)}$

g) $f(x) = \arcsin(1 - e^{3x}) + \arctan(2^x + 1)$

h) $f(x) = \arccos(x^3 - x + 1)e^{\sin\left(\sqrt{x^2 + 3}\right)}$

i) $f(x) = x^{\sqrt{x}}$

j) $f(x) = x^{x\log x}$

4. Let $f(x) = \sqrt[3]{x^2} \cdot \sin\sqrt[3]{x^2}$. Calculate $f'(x)$. (At $x = 0$ use the definition.)

5. Let $f(x) = \arctan\dfrac{1 + x}{1 - x}$ if $x \neq 1$ and $f(1) = \beta$.
a) Is it possible to choose the value of β such that f is continuous at $x = 1$?
b) Calculate $f'(x)$ if $x \neq 1$.
c) $\lim\limits_{x \to 1} f'(x) = ?$ Does $f'(1)$ exist?

6. Prove that the function $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentiable on $\mathbb{R}$ but f' is not continuous.

7. Choose the values of the parameters such that the following functions be differentiable on $\mathbb{R}$:

a) $f(x) = \begin{cases} x^2 & \text{if } x \leq x_0 \\ ax + b & \text{if } x > x_0 \end{cases}$

b) $f(x) = \begin{cases} e^{2x} & \text{if } x \geq 0 \\ ax^2 + bx + c & \text{if } x < 0 \end{cases}$

8. Find the equation of the tangent line to the following functions at x_0:

a) $f(x) = \sin \sqrt{x}$, $x_0 = \pi^2$ b) $f(x) = x^3 - 8x$, $x_0 = 3$ c) $f(x) = e^{\sin x}$, $x_0 = \pi$

9. Does the function $f(x) = x^2 - 1$ has a tangent line that passes through the point $(2, 2)$? If so, then find the equation of the tangent line.

10. Find the equation of the tangent line to the following curves at P:

a) $x^3 + y^3 - 6xy = 0$, $P(3, 3)$ b) $2xy + \pi \sin y = 2\pi$, $P\left(1, \dfrac{\pi}{2}\right)$ c) $x \sin 2y = y \cos 2x$, $P\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

L'Hospital's rule

14. Use L'Hospital's rule to evaluate the following limits:

a) $\lim\limits_{x \to 0} \dfrac{\ln(1+x)}{x}$ b) $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x^3}$ c) $\lim\limits_{x \to 0} \dfrac{3^{\sin x} - 1}{x}$ d) $\lim\limits_{x \to 0} \dfrac{x(1 - \cos x)}{x - \sin x}$

e) $\lim\limits_{x \to \infty} \dfrac{x e^{\frac{x}{2}}}{e^x + 1}$ f) $\lim\limits_{x \to \infty} \dfrac{\ln x^2}{\sqrt{x}}$ g) $\lim\limits_{x \to 0} \dfrac{\ln x}{1 + \ln \sin x}$ h) $\lim\limits_{x \to 0} (\arcsin x)(\cot x)$

i) $\lim\limits_{x \to -\infty} x^2 e^x$ j) $\lim\limits_{x \to \frac{\pi}{2}} \left(\dfrac{\pi}{2} - x\right) \tan x$ k) $\lim\limits_{x \to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right)$ l) $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x \sin x}\right)$

m) $\lim\limits_{x \to 0+} x^x$ n) $\lim\limits_{x \to 1+} x^{1/(x-1)}$ o) $\lim\limits_{x \to 0+} (e^x + x)^{\frac{1}{x}}$ p) $\lim\limits_{x \to 0+} (\sin x)^x$

q) $\lim\limits_{x \to -\infty} \dfrac{e^{8x} - 2e^{-3x}}{e^{5x} + e^{-3x}}$ r) $\lim\limits_{x \to \infty} \dfrac{\text{sh}(3x - 2)}{\text{ch}(3x + 4)}$

Mean value theorems

11. Prove that the polynomial $f(x) = x^7 + 14x + 3$ has exactly one root.

12.* Prove that the function $f(x) = x^n + ax + b$ has at most two roots if n is even and at most 3 roots if n is odd.

13. Prove the following inequalities:

a) $|\sin x - \sin y| \leq |x - y|$,   if $x < y$

b) $\dfrac{b-a}{\cos^2 a} < \tan b - \tan a < \dfrac{b-a}{\cos^2 b}$,   if $0 < a < b < \dfrac{\pi}{2}$

c) $\sin x \leq \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right) + \dfrac{1}{2}$,   if $\dfrac{\pi}{6} \leq x \leq \dfrac{\pi}{2}$

d) $\tan x - 1 > 2x - \dfrac{\pi}{2}$,   if $0 < x < \dfrac{\pi}{4}$

e) $\log(1 + x) \leq x$,   if $x \geq 0$

f) $e^x \geq 1 + x$,   if $x \geq 0$