Practice exercises 10.

1. Analyze the following functions and sketch their graphs:

a) $f(x) = 2x^4 - 4x^2 + 1$ b) $f(x) = \frac{8}{x^2 + 4}$ c) $(x) = \frac{x}{x^2 + 1}$

d) $f(x) = \frac{x}{x^2 - 1}$ e) $f(x) = \frac{x^2}{x - 1}$ f) $f(x) = x^2 \ln x^2$

g) $f(x) = x - \arctan \frac{x}{x + 1}$ h) $f(x) = (x - 3)\, e^{-x}$ i) $f(x) = (x + 2)^2\, e^{-x}$

j) $f(x) = e^{-x^2}$ k) $f(x) = x\, e^{-x^2}$ l) $f(x) = x + \sin x,\ x \in [0,\ 2\pi]$

2. Find the absolute extreme values of the following functions on the given intervals:
a) $f(x) = x^3 - 6x^2 - 15x + 3$ on $[-6,\ 6]$
b) $f(x) = x^2 \ln x$ on $[1,\ e]$

3. Write 50 as the sum of two positive numbers so that their product is maximal.

4. We make a rectangular box with a square base open at the top. The maximum surface area of the box is $A = 2\ m^2$. How should we choose the dimensions of the box so that its volume V is the largest, and what is this maximum volume?

5. Find the maximal volume of a cylinder inscribed in a sphere of radius 1.

6. Determine the dimensions of a straight cylinder of a given volume when the surface area is the least possible.

7. Find the minimal and maximal distance between the point $A(2,\ 0)$ and the points of the circle with equation $x^2 + y^2 = 1$.

8.* Find the maximal element of the sequence $a_n = \frac{n^2}{n^3 + 100}$.

9.* Prove that the polynomial

$p(x) = 1 - 2x^{11} + 3x^{24} - 4x^{35} + 5x^{46}$

has at most 4 real roots.
Hint: If a real polynomial $q(x)$ has n roots, then at least how many sign changes does q' have? Apply this several times.