Calculus 1, Final exam, Part 1

5th January, 2022

Name: ___________________________________ Neptun code: __________

I.	II.	III.	IV.	$\sum$

I. Definitions and theorems (12 x 3 points)

1. What is the relationship between boundedness, monotonicity and convergence of sequences?
2. State the root test for number series.
3. State the integral test for number series.
4. What does it mean that a series is conditionally convergent?
5. What does it mean that the number $x \in \mathbb{R}$ is a limit point of the set $A \subset \mathbb{R}$?
6. What is the sequential criterion for a limit of a function?
7. What does it mean that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous on an interval $I \subset \mathbb{R}$?
8. What is the definition of a function being differentiable at a point $x_0 \in \mathbb{R}$?
9. State Weierstrass' extreme value theorem for continuous functions.
10. State Taylor's theorem with the remainder term.
11. State the Newton-Leibniz formula.
12. Give two sufficient conditions for a function $f : [a, b] \longrightarrow \mathbb{R}$ to be Riemann integrable.

II. Proof of a theorem (15 points)

Write down the statement of Rolle's theorem and prove it.

III. True or false? (15 x 3 points)

Indicate at each statement whether it is true or false and give a short explanation for your answer.

1. If $\lim_{n\to\infty} a_n^2 = +\infty$ then $\lim_{n\to\infty} a_n = -\infty$ or $\lim_{n\to\infty} a_n = \infty$.
2. If the sequences (a_n) and $(a_n + b_n)$ are convergent then the sequence (b_n) is also convergent.
3. There exists a sequence that does not have a monotonic subsequence.

4. If a nonnegative series $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \sqrt{a_n}$ also converges.

5. If $a_n > 0$ for all $n \in \mathbb{N}$ and the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

6. The set of rational numbers is open in $\mathbb{R}$.

7. There exists a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is continuous only at one point.

8. If the function f is continuous on (a, b) then f is bounded on (a, b).

9. If the function f is continuous at x_0 then f is differentiable at x_0.

10. Let $f(x) = x^4 - x^2$ for $x \in [-1, 1]$. Then there exists a point in $(-1, 1)$ at which the tangent line is parallel to the x-axis.

11. If the function f is twice differentiable on an open interval containing x_0 and $f'(x_0) = f''(x_0) = 0$ then f has an inflection point at x_0.

12. If a function f is differentiable on $[a, b]$ and $f'(x) \geq 0$ for all $x \in [a, b]$ then it implies that $f(b) \geq f(a)$.

13. The partial fraction decomposition of $f(x) = \dfrac{1+x}{(x-1)^3 \left(x^2+x+1\right)}$ cannot contain the term $\dfrac{A}{(x-1)^4}$.

14. The function $f(x) = \operatorname{sgn}(x)\, e^x$ is Riemann integrable on the interval $[-1, 1]$.

15. The improper integral $\int_1^{\infty} \dfrac{\arctan x}{x^2}\, dx$ is convergent.

IV. Examples (3 x 3 points)

1. Give an example for a sequence (a_n) such that (a_n) is not bounded and $\dfrac{1}{a_n}$ is not convergent.

2. Give an example for a function $f : [a, b] \longrightarrow \mathbb{R}$ that does not have an antiderivative.

3. Give an example for a function $f : [1, 2) \longrightarrow \mathbb{R}$ such that f is not bounded but $\int_1^2 f(x)\, dx$ is convergent.

Answers

III. True or false? (15 x 3 points)

Indicate at each statement whether it is true or false and give a short explanation for your answer.

1. If $\lim\limits_{n\to\infty} a_n^2 = +\infty$ then $\lim\limits_{n\to\infty} a_n = -\infty$ or $\lim\limits_{n\to\infty} a_n = \infty$.

False. For example if $a_n = (-1)^n\, n$ then $\lim\limits_{n\to\infty} a_n$ doesn't exist but $\lim\limits_{n\to\infty} a_n^2 = +\infty$.

2. If the sequences (a_n) and $(a_n + b_n)$ are convergent then the sequence (b_n) is also convergent.

True. Since $b_n = (a_n + b_n) - a_n$ then (b_n) is also convergent (by the difference rule).

3. There exists a sequence that does not have a monotonic subsequence.

False. Every sequence has a monotonic subsequence by the Bolzano-Weierstrass theorem.

4. If a nonnegative series $\sum\limits_{n=1}^{\infty} a_n$ converges then $\sum\limits_{n=1}^{\infty} \sqrt{a_n}$ also converges.

False. For example if $a_n = \dfrac{1}{n^2}$ then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges but $\sum\limits_{n=1}^{\infty} \sqrt{a_n} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

5. If $a_n > 0$ for all $n \in \mathbb{N}$ and the alternating series $\sum\limits_{n=1}^{\infty} (-1)^n\, a_n$ is convergent then $\lim\limits_{n\to\infty} a_n = 0$.

True. By the $n$the term test $\lim\limits_{n\to\infty} (-1)^n\, a_n = 0 \Longrightarrow \lim\limits_{n\to\infty} a_n = 0$.

(Or, if $\lim\limits_{n\to\infty} a_n \neq 0$ then $\lim\limits_{n\to\infty} (-1)^n\, a_n$ doesn't exist, so by the $n$the term test he series would be divergent.)

6. The set of rational numbers is open in $\mathbb{R}$.

False. If $x \in \mathbb{Q}$ and I is an interval containing x then I contains irrational numbers, so I is not a subset of $\mathbb{Q}$.

7. There exists a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is continuous only at one point.

True. For example let $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$. Then f is continuous only at 0.

8. If the function f is continuous on (a, b) then f is bounded on (a, b).

False. For example $f(x) = \tan x$ is continuous on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ but not bounded.

9. If the function f is continuous at x_0 then f is differentiable at x_0.

False. For example $f(x) = |x|$ is continuous at 0 but not differentiable at 0.

10. Let $f(x) = x^4 - x^2$ for $x \in [-1, 1]$. Then there exists a point in $(-1, 1)$ at which the tangent line is

parallel to the x-axis.

True. Since f is differentiable on $[-1, 1]$ and $f(1) = f(-1)$ then by Rolle's theorem there exists $c \in (-1, 1)$ such that $f'(c) = 0$.

11. If the function f is twice differentiable on an open interval containing x_0 and $f'(x_0) = f''(x_0) = 0$ then f has an inflection point at x_0.

False. For example if $f(x) = x^4$ then $f'(0) = f''(0) = 0$ but f has a local minimum at $0 \Longrightarrow f$ doesn't have an inflection point at 0.

12. If a function f is differentiable on $[a, b]$ and $f'(x) \geq 0$ for all $x \in [a, b]$ then it implies that $f(b) \geq f(a)$.

True. By Lagrange's mean value theorem there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \Longrightarrow f(b) - f(a) = f'(c)(b - a) \geq 0 \Longrightarrow f(b) \geq f(a)$$

13. The partial fraction decomposition of $f(x) = \dfrac{1 + x}{(x - 1)^3 \left(x^2 + x + 1\right)}$ cannot contain the term $\dfrac{A}{(x - 1)^4}$.

True. The partial fraction decomposition of f is

$$f(x) = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3} + \frac{D x + E}{x^2 + x + 1}$$

14. The function $f(x) = \operatorname{sgn}(x)\, e^x$ is Riemann integrable on the interval $[-1, 1]$.

True, since f is bounded on $[-1, 1]$ and has a jump discontinuity at 0.

15. The improper integral $\displaystyle\int_1^\infty \frac{\arctan x}{x^2}\, dx$ is convergent.

True. Since $0 < \displaystyle\int_1^\infty \frac{\arctan x}{x^2}\, dx < \int_1^\infty \frac{\pi}{2} \frac{1}{x^2}\, dx$ and $\displaystyle\int_1^\infty \frac{1}{x^2}\, dx$ converges then by the comparison test for improper integrals $\displaystyle\int_1^\infty \frac{\arctan x}{x^2}\, dx$ also converges.

IV. Examples (3 x 3 points)

1. Give an example for a sequence (a_n) such that (a_n) is not bounded and $\dfrac{1}{a_n}$ is not convergent.

For example let $a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases} \Longrightarrow (a_n)$ is not bounded.

Then $\dfrac{1}{a_n} = \begin{cases} 1 \longrightarrow 1 & \text{if } n \text{ is odd} \\ \dfrac{1}{n} \longrightarrow 0 & \text{if } n \text{ is even} \end{cases} \Longrightarrow \displaystyle\lim_{n \to \infty} \frac{1}{a_n}$ doesn't exist.

2. Give an example for a function $f:[a, b] \longrightarrow \mathbb{R}$ that does not have an antiderivative.

Any function with a jump discontinuity is suitable. For example $f(x) = \operatorname{sgn}(x)$ on $[-1, 1]$ doesn't have an antiderivative by Darboux's theorem.
Or, the Dirichlet function on $[-1, 1]$ doesn't have an antiderivative either.

3. Give an example for a function $f:[1, 2) \longrightarrow \mathbb{R}$ such that f is not bounded but $\int_1^2 f(x)\, dx$ is convergent.

We know that $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ converges $\left(\text{and } \int_0^1 \frac{1}{\sqrt{x}}\, dx = 2\right)$ so we can transform the integral:
$\int_1^2 \frac{1}{\sqrt{2-x}}\, dx$ also converges $\left(\text{and } \int_1^2 \frac{1}{\sqrt{2-x}}\, dx = 2\right)$.