Calculus 1, Final exam, Part 1

10th January, 2023

Name: ________________________________ Neptun code: ________________

Part I: ________ Part II.: ________ Part III.: ________ Sum: ________

I. Definitions and theorems (15 x 3 points)

1. What does it mean that $\lim_{n\to\infty} a_n = -\infty$?
2. State the Bolzano-Weierstrass theorem for number sequences.
3. Define the limes superior of the sequence (a_n).
4. State the nth term test for number series.
5. State the ratio test for number series.
6. What does it mean that the number $x \in \mathbb{R}$ is an interior point of the set $A \subset \mathbb{R}$?
7. State the sequential criterion for continuity.
8. What does it mean that a function has a jump discontinuity?
9. Give a sufficient condition for a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ to be uniformly continuous on the set $A \subset \mathbb{R}$.
10. State Lagrange's mean value theorem.
11. What does it mean that a function is concave? Write down a necessary and sufficient condition for a function to be concave on an interval.
12. State the L'Hospital's rule.
13. Give two sufficient conditions for a function f to have a local maximum at the point x_0.
14. State the integration-by-parts formula.
15. What is the formula for the arc-length of a function $y = f(x)$, $x \in [a, b]$?

II. Proof of a theorem (15 points)

Write down statement of the Newton-Leibniz formula and prove it.

III. True or false? (15 x 3 points)

Indicate at each statement whether it is true or false and give a short explanation for your answer. The correct answer without an explanation is worth 1 point.

1. $\lim_{n\to\infty} a_n = A$ if and only if for all $\varepsilon > 0$ the sequence (a_n) has infinitely many terms in the interval $(A - \varepsilon, A + \varepsilon)$.
2. If for all $K > 0$ the sequence (a_n) has only finitely many terms outside the interval (K, ∞) then $\lim_{n\to\infty} a_n = \infty$.

3. If $\lim_{n\to\infty} a_n = \frac{1}{2}$ then $\lim_{n\to\infty} a_n^n = 0$.

4. If $\sum_{n=1}^{\infty} a_n$ is convergent and $b_n < a_n$ for all positive integer n then $\sum_{n=1}^{\infty} b_n$ is also convergent.

5. If the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

6. If x is a boundary point of $A \subset \mathbb{R}$ then x is a limit point of A.

7. The polynomial $f(x) = -x^7 + 10x^4 + 3x + 10$ has at least one real root.

8. There exists a continuous function $f : [-1, 1] \longrightarrow \mathbb{R}$ such that f has neither a minimum nor a maximum on $[-1, 1]$.

9. There exists a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f'(x) = 2$ if $x \geq 0$ and $f'(x) = -2$ if $x < 0$.

10. The function $f(x) = 2x + \sin x$ is invertible on $\mathbb{R}$.

11. If a function is differentiable everywhere on $\mathbb{R}$ and $|f'(x_0)| \leq 1$ for all $x_0 \in [4, 5]$ then $|f(4) - f(5)| \leq 1$.

12. Assume that f is at least two times differentiable on $\mathbb{R}$. If f has a local minimum at x_0 then $f'(x_0) = 0$ and $f''(x_0) > 0$.

13. Let $f(x) = \frac{1}{x}$ if $0 < x < 1$ and $f(0) = 0$. Then f is Riemann integrable on $[0, 1]$.

14. The function $f(x) = \arctan\left(\frac{x^2}{x+3}\right)$ is Riemann integrable on the interval $[0, 10]$.

15. If f is Riemann integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, where $x \in [a, b]$, then F is uniformly continuous on $[a, b]$.

Answers

I. Definitions and theorems (15 x 3 points)

1. What does it mean that $\lim_{n\to\infty} a_n = -\infty$?

Page 4: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-04-05.pdf

2. State the Bolzano-Weierstrass theorem for number sequences.

Page 1: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-08-09.pdf

3. Define the limes superior of the sequence (a_n).

Page 4: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-08-09.pdf

4. State the nth term test for number series.

Page 9: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-08-09.pdf

5. State the ratio test for number series.

Page 7: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-10-11.pdf

6. What does it mean that the number $x \in \mathbb{R}$ is an interior point of the set $A \subset \mathbb{R}$?

Page 2: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-12-13.pdf

7. State the sequential criterion for continuity.

Page 9: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-14.pdf

8. What does it mean that a function has a jump discontinuity?

Page 12: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-14.pdf

9. Give a sufficient condition for a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ to be uniformly continuous on the set $A \subset \mathbb{R}$.

Page 7 (Heine's theorem) or Page 8 (Lipschitz continuity): https://math.bme.hu/~nagyi/calculus1-2022/calculus1-15-16.pdf

10. State Lagrange's mean value theorem.

Page 13: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-17-18.pdf

11. What does it mean that a function is concave? Write down a necessary and sufficient condition for a function to be concave on an interval.

Definition: page 10: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-15-16.pdf
Condition: page 11: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-19-20.pdf

12. State the L'Hospital's rule.

Page 1: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-19-20.pdf

13. Give two sufficient conditions for a function f to have a local maximum at the point x_0.

Page 7, first derivative test and second derivative test: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-19-20.pdf

14. State the integration-by-parts formula.

Page 3: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-21.pdf

15. What is the formula for the arc-length of a function $y = f(x)$, $x \in [a, b]$?

Page 10: https://math.bme.hu/~nagyi/calculus1-2022/calculus1-22-23.pdf

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III. True or false? (15 x 3 points)

1. $\lim_{n\to\infty} a_n = A$ if and only if for all $\varepsilon > 0$ the sequence (a_n) has infinitely many terms in the interval $(A - \varepsilon, A + \varepsilon)$.

False. For example $a_n = (-1)^n$ is divergent and there are infinitely many terms in the interval $(0, 2)$.

2. If for all $K > 0$ the sequence (a_n) has only finitely many terms outside the interval (K, ∞) then $\lim_{n\to\infty} a_n = \infty$.

True. The statement is equivalent with the definition: $\lim_{n\to\infty} a_n = \infty \iff$ for all $K > 0$ there exists $N \in \mathbb{N}$ such that $a_n > K$ if $n > N$.

3. If $\lim_{n\to\infty} a_n = \frac{1}{2}$ then $\lim_{n\to\infty} a_n^n = 0$.

True. If n is large enough then $0 < a_n < \frac{3}{4}$, so $0 < a_n^n < \left(\frac{3}{4}\right)^n \longrightarrow 0$, and thus by the sandwich theorem $\lim_{n\to\infty} a_n^n = 0$.

4. If $\sum_{n=1}^{\infty} a_n$ is convergent and $b_n < a_n$ for all positive integer n then $\sum_{n=1}^{\infty} b_n$ is also convergent.

False. For example if $a_n = \frac{1}{n^2}$ and $b_n = -1$ then $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} b_n$ is divergent.

5. If the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

True. By the nth term test $\lim_{n\to\infty} (-1)^n a_n = 0 \implies \lim_{n\to\infty} a_n = 0$.

6. If x is a boundary point of $A \subset \mathbb{R}$ then x is a limit point of A.

False. A boundary point can be an isolated point. For example, if $A = (0, 1) \cup \{2\}$ then $x = 2$ is a boundary point but not a limit point of A.

7. The polynomial $f(x) = -x^7 + 10x^4 + 3x + 10$ has at least one real root.

True. f is continuous and since $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$ then there exists an interval $[a, b]$ such that $f(a) > 1$ and $f(b) < -1$. So by Bolzano's theorem there exists $x \in (a, b)$ such that $f(x) = 0$.

8. There exists a continuous function $f : [-1, 1] \longrightarrow \mathbb{R}$ such that f has neither a minimum nor a maximum on $[-1, 1]$.

False. By Weierstrass extreme value theorem if a function is continuous on a closed interval then it has both a minimum and a maximum on the interval.

9. There exists a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f'(x) = 2$ if $x \geq 0$ and $f'(x) = -2$ if $x < 0$.

False. By Darboux' theorem f' cannot have a jump discontinuity.

10. The function $f(x) = 2x + \sin x$ is invertible on $\mathbb{R}$.

True. Since $f'(x) = 2 + \cos x > 0$ for all $x \in \mathbb{R}$ then f is strictly monotonically increasing on $\mathbb{R}$, so f is invertible on $\mathbb{R}$.

11. If a function is differentiable everywhere on $\mathbb{R}$ and $|f'(x_0)| \leq 1$ for all $x_0 \in [4, 5]$ then $|f(5) - f(4)| \leq 1$.

True. By Lagrange's theorem there exists $c \in (4, 5)$ such that $f'(c) = \frac{f(5) - f(4)}{5 - 4}$. Since $|f'(x_0)| \leq 1$ for all $x_0 \in [4, 5]$ then $|f(5) - f(4)| \leq 1$.

12. Assume that f is at least two times differentiable on $\mathbb{R}$. If f has a local minimum at x_0 then $f'(x_0) = 0$ and $f''(x_0) > 0$.

False. For example $f(x) = x^4$ has a local minimum at $x_0 = 0$, $f'(0) = 0$, but $f''(0) = 0$.

13. Let $f(x) = \frac{1}{x}$ if $0 < x < 1$ and $f(0) = 0$. Then f is Riemann integrable on $[0, 1]$.

False. f is not bounded, but boundedness is necessary for Riemann integrability.

14. The function $f(x) = \arctan\left(\frac{x^2}{x+3}\right)$ is Riemann integrable on the interval $[0, 10]$.

True. Since f is a composition of continuous functions then it is continuous, so it is Riemann integrable on $[0, 10]$.

15. If f is Riemann integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, where $x \in [a, b]$, then F is uniformly continuous on $[a, b]$.

True. By the second fundamental theorem of calculus, F is Lipschitz continuous on $[a, b]$, so f is uniformly continuous on $[a, b]$.