Calculus 1, Final exam, Part 1

12th January, 2022

Name: ________________________________ Neptun code: _________

I.	II.	III.	IV.	$\sum$

I. Definitions and theorems (12 x 3 points)

1. What is the statement of the sandwich theorem for number sequences?
2. What does it mean that the sequence (a_n) is a Cauchy sequence?
3. State Leibniz's theorem for alternating series.
4. What are the possible behaviours of a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ with regard to the region of convergence?
5. What does it mean that the number $x \in \mathbb{R}$ is a boundary point of the set $A \subset \mathbb{R}$?
6. State the intermediate value theorem or Bolzano's theorem.
7. What does it mean that a function f has an essential discontinuity at x_0?
8. What does it mean that a function is convex? Write down a necessary and sufficient condition for a function to be convex on an interval.
9. State the theorem about the derivative of the inverse of a function.
10. State the L'Hospital's rule.
11. Give two sufficient conditions for a function f to have an inflection point at x_0.
12. State the second fundamental theorem of calculus.

II. Proof of a theorem (15 points)

Write down the statement of Lagrange's theorem and prove it.

III. True or false? (15 x 3 points)

Indicate at each statement whether it is true or false and give a short explanation for your answer.

1. If $a_n < 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = -\infty$ then $\lim_{n\to\infty} a_n b_n = +\infty$.
2. If $\lim_{n\to\infty} a_n = A \in \mathbb{R}$ then $a_n > A + 1$ can occur only finitely many times.

3. If a sequence is strictly monotonically increasing then it is divergent.

4. If the sequence (a_n) is monotonically decreasing and $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

5. For all $A \subset \mathbb{R}$ if x is an accumulation point of A then $x \in A$.

6. If the function f is continuous on (a, b) then f has a minimum and a maximum on (a, b).

7. There exists a function $f : [a, b] \longrightarrow \mathbb{R}$ that is Lipschitz continuous but not uniformly continuous.

8. If the function f is differentiable at x_0 and the function g is not differentiable at x_0 then $f + g$ is not differentiable at x_0.

9. If the function f is differentiable on $[a, b]$ and f has exactly three roots in $[a, b]$ then f' has at least two roots in $[a, b]$.

10. If a function f is differentiable everywhere on $\mathbb{R}$ and $|f(5) - f(6)| \leq 1$ then $|f'(x)| \leq 1$ for some $x \in [5, 6]$.

11. If the function $f : (a, b) \longrightarrow \mathbb{R}$ is differentiable and strictly monotonically increasing then $f'(x) > 0$ for all $x \in (a, b)$.

12. There exists a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f'(x) = \operatorname{sgn}(x)$ for all $x \in \mathbb{R}$.

13. If the function f is continuous on $[a, b]$ then f has an antiderivative on (a, b).

14. The function $f(x) = \sin\left(\frac{e^x}{x}\right)$ is Riemann integrable on $[1, 2]$.

15. The improper integral $\int_1^{\infty} \frac{2 + \sin x}{x^3}\, dx$ is convergent.

IV. Examples (3 x 3 points)

1. Give an example for a nonnegative series $\sum_{n=0}^{\infty} a_n$ such that $\sqrt[n]{a_n} < 1$ for all n, $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$ and $\sum_{n=0}^{\infty} a_n$ does not converge.

2. Give an example for a function $f : [a, b] \longrightarrow \mathbb{R}$ that is not Riemann integrable.

3. Give an example for a function $f : [0, \infty) \longrightarrow \mathbb{R}$ such that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Solutions

III. True or false? (15 x 3 points)

Indicate at each statement whether it is true or false and give a short explanation for your answer.

1. If $a_n < 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = -\infty$ then $\lim_{n\to\infty} a_n b_n = +\infty$.

False. For example $a_n = -\frac{1}{n} < 0$ and $b_n = -n \longrightarrow -\infty$ but $a_n b_n = 1 \longrightarrow 1$.

2. If $\lim_{n\to\infty} a_n = A \in \mathbb{R}$ then $a_n > A + 1$ can occur only finitely many times.

True. By the definition of the limit for $\varepsilon = 1$ there exists $N \in \mathbb{N}$ such that for all $n > N$: $A - 1 < a_n < A + 1 \implies a_n > A + 1$ can occur only finitely many times (at most N times).

3. If a sequence is strictly monotonically increasing then it is divergent.

False. For example $a_n = 1 - \frac{1}{n}$ is monotonically increasing but $a_n \longrightarrow 1$.

4. If the sequence (a_n) is monotonically decreasing and $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

False. For example $a_n = \frac{1}{n}$ is monotonically decreasing and $a_n \longrightarrow 0$ but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

5. For all $A \subset \mathbb{R}$ if x is an accumulation point of A then $x \in A$.

False. For example if $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ or $A = (0, 1)$ then $x = 0$ is an accumulation point of A but $x \notin A$.

6. If the function f is continuous on (a, b) then f has a minimum and a maximum on (a, b).

False. For example $f(x) = \tan x$ is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ but f has no maximum and no minimum.

7. There exists a function $f : [a, b] \longrightarrow \mathbb{R}$ that is Lipschitz continuous but not uniformly continuous.

False. If a function is Lipschitz continuous then it is uniformly continuous.

8. If the function f is differentiable at x_0 and the function g is not differentiable at x_0 then $f + g$ is not differentiable at x_0.

True. $g = (f + g) - f \implies$ if $(f + g)$ and f are differentiable then g is also differentiable.

9. If the function f is differentiable on $[a, b]$ and f has exactly three roots in $[a, b]$ then f' has at least two roots in $[a, b]$.

True. If the roots are x, y, z and $a \le x < y < z \le b$ then by Rolle's theorem there exist $x < c_1 < y$ and $y < c_2 < z$ such that $f'(c_1) = 0$ and $f'(c_2) = 0$.

10. If a function f is differentiable everywhere on $\mathbb{R}$ and $|f(5) - f(6)| \le 1$ then $|f'(x)| \le 1$ for some $x \in [5, 6]$.

True. By Lagrange's theorem there exists $c \in (5, 6)$ such that $\frac{f(6) - f(5)}{6 - 5} = f'(c) \implies$

$|f(5) - f(6)| = |f'(c)| \le 1$.

11. If the function $f : (a, b) \longrightarrow \mathbb{R}$ is differentiable and strictly monotonically increasing then $f'(x) > 0$ for all $x \in (a, b)$.

False. For example $f(x) = x^3$ is differentiable and strictly monotonically increasing on $(-1, 1)$ but $f'(0) = 0$. (However, the converse of the statement is true.)

12. There exists a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f'(x) = \operatorname{sgn}(x)$ for all $x \in \mathbb{R}$.

False. By Darboux's theorem if $-1 < y < 1$ then $f'(x) = y$ should hold for some x but if $y = \frac{1}{2}$ then there is no such x.

13. If the function f is continuous on $[a, b]$ then f has an antiderivative on (a, b).

True. It is a consequence of the second fundamental theorem of calculus: If f is continuous at $x_0 \in [a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is differentiable x_0 and $F'(x_0) = f(x_0)$.

14. The function $f(x) = \sin\left(\frac{e^x}{x}\right)$ is Riemann integrable on $[1, 2]$.

True. Since f is a composition of continuous functions then it is continuous, so it is Riemann integrable on $[1, 2]$.

15. The improper integral $\int_1^\infty \frac{2 + \sin x}{x^3}\,dx$ is convergent.

True. Since $0 < \int_1^\infty \frac{2 + \sin x}{x^3}\,dx < \int_1^\infty \frac{3}{x^3}\,dx$ and $\int_1^\infty \frac{3}{x^3}\,dx$ converges then by the comparison test for improper integrals $\int_1^\infty \frac{2 + \sin x}{x^3}\,dx$ also converges.

IV. Examples (3 x 3 points)

1. Give an example for a nonnegative series $\sum_{n=0}^\infty a_n$ such that $\sqrt[n]{a_n} < 1$ for all n, $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$ and $\sum_{n=0}^\infty a_n$ does not converge.

For example if $a_n = \frac{1}{n}$ then $\sum_{n=0}^\infty \frac{1}{n}$ diverges, $\sqrt[n]{a_n} = \frac{1}{\sqrt[n]{n}} < 1$ and $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$.

2. Give an example for a function $f : [a, b] \longrightarrow \mathbb{R}$ that is not Riemann integrable.

For example the Dirichlet function $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ is not Riemann integrable on any closed interval, since the lower Darboux integral is 0 and the upper Darboux integral is $b - a$.

3. Give an example for a function $f : [0, \infty) \longrightarrow \mathbb{R}$ such that $\int_{-\infty}^\infty f(x)\,dx = 1$.

For example:

1) $\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty} [-e^{-x}]_0^A = \lim_{A\to\infty} \left(-e^{-A} + e^0\right) = 0 + 1 = 1 \implies \frac{1}{2} \int_{-\infty}^\infty e^{-|x|}\,dx = 1$

2) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{\substack{A\to\infty \\ B\to-\infty}} [\arctan]_B^A = \lim_{\substack{A\to\infty \\ B\to-\infty}} (\arctan A - \arctan B) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi \implies \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 1$