Calculus 1, Final exam 2, Part 1

16th January, 2023

Name: ________________________________ Neptun code: ________________

Part I: ________ Part II.: ________ Part III.: ________ Sum: ________

I. Definitions and theorems (15 x 3 points)

1. What does it mean that $\lim_{n\to\infty} a_n = A$, where $A \in \mathbb{R}$?
2. State the sandwich theorem for number sequences.
3. State the root test for number series.
4. State Leibniz's theorem for alternating series.
5. What does it mean that the number $x \in \mathbb{R}$ is a boundary point of the set $A \subset \mathbb{R}$?
6. What does it mean that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at the point $x_0 \in \mathbb{R}$?
7. What does it mean that a function f has a removable discontinuity at the point $x_0 \in \mathbb{R}$?
8. State the intermediate value theorem or Bolzano's theorem.
9. State Weierstrass' extreme value theorem for continuous functions.
10. What does it mean that a function is convex? Write down the definition.
11. What does it mean that a function is differentiable at the point $x_0 \in \mathbb{R}$?
12. State Darboux's theorem.
13. Give two sufficient conditions for a function to have an inflection point at the point x_0.
14. Give two sufficient conditions for a function $f : [a, b] \longrightarrow \mathbb{R}$ to be Riemann integrable.
15. State the Newton-Leibniz formula.

II. Proof of a theorem (15 points)

Write down the statement of Rolle's theorem and prove it.

III. True or false? (15 x 3 points)

Indicate at each statement whether it is true or false and give a short explanation for your answer. The correct answer without an explanation is worth 1 point.

1. $\lim_{n\to\infty} a_n = A$ if and only if for all $\varepsilon > 0$ the sequence (a_n) has only finitely many terms outside of the interval $(A - \varepsilon, A + \varepsilon)$.
2. If the sequence (a_n) is bounded then it has a smallest and a greatest term.
3. If the sequence (a_n) is bounded below but isn't bounded above then $\lim_{n\to\infty} a_n = \infty$.
4. If $\lim_{n\to\infty} a_n = 10^{-10}$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

5. If the sequence (a_n) is monotonically decreasing and $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

6. If $x_0 \in \mathbb{R}$ is an interior point of $A \in \mathbb{R}$ then x_0 is a limit point of A.

7. If the function f is continuous on (a, b) then f is bounded.

8. If the function f is continuous on $(-1, 1)$, then f has a minimum and a maximum on $(-1, 1)$.

9. The function $f(x) = x^3 - \sqrt{x^2 + 5}$ does not have a root on the interval $[0, 2]$.

10. Let $f(2023) = 1$ and $f(x) = 0$, if $x \neq 2023$. Then f is differentiable at $x = 2023$ from the right and from the left.

11. Let $f(x) = | x - 2023 |$. Then f is differentiable at $x = 2023$ from the right and from the left.

12. Let $f(x) = 2x^3 - x$ for $x \in [0, 1]$. Then there exists a point in $x_0 \in (0, 1)$ such that the tangent line of the graph of f at $(x_0, f(x_0))$ is parallel to the straight line $y = x$.

13. If the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is even, then f' is odd.

14. The partial fraction decomposition of $f(x) = \frac{1}{x^5 - x^3}$ cannot contain the term $\frac{1}{(x-1)^2}$.

15. If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and $F(x) = \int_a^x f(t)\,dt$, where $x \in [a, b]$, then $F'(x) = f(x)$.

Answers

I. Definitions and theorems (15 x 3 points)

1. What does it mean that $\lim_{n\to\infty} a_n = A$, where $A \in \mathbb{R}$?

Definition: $\lim_{n\to\infty} a_n = A \in \mathbb{R} \iff$ for all $\varepsilon > 0$ there exists a threshold index $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$, $| a_n - A| < \varepsilon$.

2. State the sandwich theorem for number sequences.

Theorem: If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$, $c_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ and $a_n \leq b_n \leq c_n$ for all $n > N$, then $b_n \xrightarrow{n\to\infty} A \in \mathbb{R}$

3. State the root test for number series.

Theorem: Assume that $a_n > 0$ and $\limsup \sqrt[n]{a_n} = R$. Then

(1) if $R < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent;

(2) if $R > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

4. State Leibniz's theorem for alternating series.

Theorem: Let (a_n) be a monotonically decreasing sequence of positive numbers such that $a_n \xrightarrow{n\to\infty} 0$. Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots$ is convergent.

5. What does it mean that the number $x \in \mathbb{R}$ is a boundary point of the set $A \subset \mathbb{R}$?

Definition: $x \in \mathbb{R}$ is a boundary point of $A \subset \mathbb{R}$, if for all $r > 0$: $B(x, r) \cap A \neq \emptyset$ and $B(x, r) \cap (\mathbb{R} \setminus A) \neq \emptyset$.
(That is, any interval $(x - r, x + r)$ contains a point in A and a point not in A.)

6. What does it mean that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at the point $x_0 \in \mathbb{R}$?

Definition: The function $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at the point $x_0 \in D_f$ if for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in D_f$ and $| x - x_0 | < \delta(\varepsilon)$ then $| f(x) - f(x_0) | < \varepsilon$.

7. What does it mean that a function f has a removable discontinuity at the point $x_0 \in \mathbb{R}$?

Definition: f has a removable discontinuity at x_0 if $\exists \lim_{x\to x_0} f(x) \in \mathbb{R}$ but $\lim_{x\to x_0} f(x) \neq f(x_0)$ or $f(x_0)$ is not defined.

8. State the intermediate value theorem or Bolzano's theorem.

Theorem: Assume that f is continuous on $[a, b]$, $f(a) \neq f(b)$ and $f(a) < c < f(b)$ or $f(b) < c < f(a)$.
Then there exists $x_0 \in (a, b)$ such that $f(x_0) = c$.

9. State Weierstrass' extreme value theorem for continuous functions.

Theorem: If f is continuous on the closed interval $[a, b]$ then there exist numbers $\alpha \in [a, b]$ and $\beta \in [a, b]$, such that $f(\alpha) \leq f(x) \leq f(\beta)$ for all $x \in [a, b]$,
that is, f has both a minimum and a maximum on $[a, b]$.

10. What does it mean that a function is convex? Write down the definition.

Definition: The function f is convex on the interval $I \subset D_f$ if for all $x, y \in I$ and $t \in [0, 1]$
$f(t x + (1 - t) y) \leq t f(x) + (1 - t) f(y)$

Or:
Definition: Let $h_{a,b}(x)$ denote the the secant line passing through the points $(a, f(a))$ and $(b, f(b))$.
The function f is convex on the interval $I \subset D_f$ if for all $\forall\, a, b \in I$ and $a < x < b \implies f(x) \leq h_{a,b}(x)$,
that is, the secant lines of f always lie above the graph of f.

11. What does it mean that a function is differentiable at the point $x_0 \in \mathbb{R}$?

Definition: Suppose that x_0 is an interior point of D_f. Then the function f is differentiable at x_0 if the following finite limit exists: $f'(x_0) = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = \lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$.

12. State Darboux's theorem.

Theorem: Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is differentiable and $f'(a) < y < f'(b)$ or $f'(b) < y < f'(a)$.
Then there exists $c \in (a, b)$ such that $f'(c) = y$.

13. Give two sufficient conditions for a function to have an inflection point at the point x_0.

Theorems:
1) If f is twice differentiable in a neighbourhood of x_0, $f''(x_0) = 0$ and f'' changes sign at x_0, then f has an inflection point at x_0.
2) If f is three times differentiable in a neighbourhood of x_0, $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then f has an inflection point at x_0.

14. Give two sufficient conditions for a function $f : [a, b] \longrightarrow \mathbb{R}$ to be Riemann integrable.

Theorems:
1) If f is monotonic and bounded on $[a, b]$ then f is Riemann integrable on $[a, b]$.
2) If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous then f is Riemann integrable on $[a, b]$.
3) If $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and continuous except finitely many points then f is Riemann integrable on $[a, b]$.

Any two conditions are suitable.

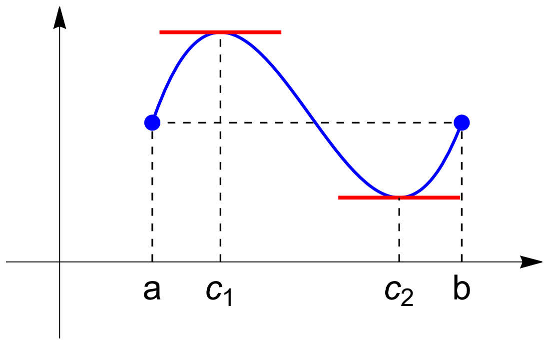
15. State the Newton-Leibniz formula.

Theorem: If $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable and $F : [a, b] \longrightarrow \mathbb{R}$ is an antiderivative of f, that is, $F'(x) = f(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\, dx = F(b) - F(a) = [F(x)]_a^b$.

II. Proof of a theorem (15 points)

Write down the statement of Rolle's theorem and prove it.

Theorem (Rolle). Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.



Proof. Since f is continuous on the closed and bounded interval $[a, b]$ then by the Weierstrass extreme value theorem f has a minimum and a maximum on $[a, b]$.

1) If both extreme values are attained at the endpoints, then
$f(x) = f(a) = f(b)$ for all $x \in [a, b] \implies f$ is constant
$\implies f'(c) = 0$ for all $c \in (a, b)$.

2) If the minimum or the maximum is attained at an interior point $c \in (a, b)$, then f has a local extremum at c, so $f'(c) = 0$.

III. True or false? (15 x 3 points)

1. $\lim_{n\to\infty} a_n = A$ if and only if for all $\varepsilon > 0$ the sequence (a_n) has only finitely many terms outside of the interval $(A - \varepsilon, A + \varepsilon)$.

True. $\lim_{n\to\infty} a_n = A \iff$ for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|a_n - A| < \varepsilon$. It is equivalent with the above statement.

2. If the sequence (a_n) is bounded then it has a smallest and a greatest term.

False. For example $a_n = \dfrac{1}{n}$ is bounded ($0 < a_n \le 1$ for all n), its greatest term is 1, but it doesn't have a smallest term.

3. If the sequence (a_n) is bounded below but isn't bounded above then $\lim_{n\to\infty} a_n = \infty$.

False. For example, let $a_{2n-1} = 0$ and $a_{2n} = n$. Then (a_n) is bounded below but isn't bounded above. Since $a_{2n-1} \longrightarrow 0$ and $a_{2n} \longrightarrow \infty$ then the limit points of (a_n) are 0 and ∞, so the limit of (a_n) doesn't exist.

4. If $\lim_{n\to\infty} a_n = 10^{-10}$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

True. Since $\lim_{n\to\infty} a_n \neq 0$ then by the nth term test $\sum_{n=1}^{\infty} a_n$ is divergent.

5. If the sequence (a_n) is monotonically decreasing and tends to zero, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

False. For example, $a_n = \frac{1}{n}$ is monotonically decreasing and $\lim_{n\to\infty} a_n = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

6. If $x_0 \in \mathbb{R}$ is an interior point of $A \in \mathbb{R}$ then x_0 is a limit point of A.

True. If x_0 is an interior point of A then there exists $r > 0$ such that $(x_0 - r, x_0 + r) \subset A$. From this it follows that any open interval $(x_0 - s, x_0 + s)$ also contains a point in $(x_0 - r, x_0 + r)$ that is distinct from x_0, so x_0 is a limit point of A.

7. If the function f is continuous on (a, b) then f is bounded.

False. For example, $f(x) = \tan x$ is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ but not bounded.

8. If the function f is continuous on $(-1, 1)$, then f has a minimum and a maximum on $(-1, 1)$.

False. For example, $f : (-1, 1) \longrightarrow \mathbb{R}$, $f(x) = x$ doesn't have a minimum and a maximum on $(-1, 1)$.

9. The function $f(x) = x^3 - \sqrt{x^2 + 5}$ does not have a root on the interval $[0, 2]$.

False. $f(2) = 5 > 0$ and $f(0) = -\sqrt{5} < 0$, so by Bolzano's theorem f has a real root on the interval $[0, 2]$. That is, there exists a number $x_0 \in (0, 2)$ such that $f(x_0) = 0$.

10. Let $f(2023) = 1$ and $f(x) = 0$, if $x \neq 2023$. Then f is differentiable at $x = 2023$ from the right and from the left.

False. Since f is not continuous at $x = 2023$ then f is not differentiable at $x = 2023$.

11. Let $f(x) = |x - 2023|$. Then f is differentiable at $x = 2023$ from the right and from the left.

True. The right-hand derivative of f at $x = 2023$ is

$$f_+'(2023) = \lim_{x\to 2023+0} \frac{f(x) - f(2023)}{x - 2023} = \lim_{x\to 2023+0} \frac{(x - 2023) - 0}{x - 2023} = 1,$$

so f is differentiable at $x = 2023$ from the right.
Similarly, $f_-'(2023) = -1$, so f is differentiable at $x = 2023$ from the left.
Remark: Since $f_+'(2023) = 1 \neq -1 = f_-'(2023)$ then f is not differentiable at $x = 2023$.

12. Let $f(x) = 2x^3 - x$ for $x \in [0, 1]$. Then there exists a point in $x_0 \in (0, 1)$ such that the tangent line of the graph of f at $(x_0, f(x_0))$ is parallel to the straight line $y = x$.

True. Since f is continuous on $[0, 1]$ and differentiable on $(0, 1)$ then by Lagrange's mean value theorem there exists $x_0 \in (0, 1)$ such that $f'(x_0) = \frac{f(1) - f(0)}{1 - 0} = 1$, which is equal to the slope of the straight line $y = x$.

13. If the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is even, then f' is odd.

True. Since f is even, then $f(x) = f(-x)$, so $f'(x) = f'(-x) \cdot (-1) = -f'(-x)$, therefore f' is odd.

14. The partial fraction decomposition of $f(x) = \dfrac{1}{x^5 - x^3}$ cannot contain the term $\dfrac{1}{(x-1)^2}$.

True. The partial fraction decomposition of $f(x) = \dfrac{1}{x^5 - x^3} = \dfrac{1}{x^3(x^2 - 1)} = \dfrac{1}{x^3(x-1)(x+1)}$ is

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^2} + \frac{D}{x-1} + \frac{E}{x+1}$$

15. If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and $F(x) = \int_a^x f(t)\, dt$, where $x \in [a, b]$, then $F'(x) = f(x)$.

True. This is the second fundamental theorem of calculus.