Calculus 1, Final exam 3, Part 1

23rd January, 2023

Name: ________________________________ **Neptun code: ________________**

Part I: ________ **Part II.: ________** **Part III.: ________** **Sum: ________**

I. Definitions and theorems (15 x 3 points)

1. What does it mean that the sequence (a_n) is a Cauchy sequence?
2. Define the limes inferior of the sequence (a_n).
3. State the ratio test for number series.
4. State the comparison test for number series.
5. What does it mean that the number $x \in \mathbb{R}$ is a limit point of the set $A \subset \mathbb{R}$?
6. What does it mean that the limit of the function f at $x_0 \in \mathbb{R}$ is $A \in \mathbb{R}$?
7. State the sequential criterion for continuity.
8. What does it mean that a function f has an essential discontinuity at $x_0 \in \mathbb{R}$?
9. State Weierstrass' extreme value theorem for continuous functions.
10. What does it mean that a function is concave? Write down the definition.
11. State the theorem about the derivative of the inverse of a function.
12. State the L'Hospital's rule.
13. Give two sufficient conditions for a function to have a local minimum at the point x_0.
14. State the Newton-Leibniz formula.
15. State the second fundamental theorem of calculus.

II. Proof of a theorem (15 points)

Write down the statement of Lagrange's mean value theorem and prove it.

III. True or false? (15 x 3 points)

Indicate at each statement whether it is true or false and give a short explanation for your answer. The correct answer without an explanation is worth 1 point.

1. If a sequence is monotonic and bounded then it has only one real limit point.
2. If (a_n) is a nonnegative sequence and $\lim_{n\to\infty} a_n = 1$ then $\lim_{n\to\infty} a_n^n = 1$.
3. If the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.
4. If the series $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

5. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

6. The set $H = [0, 1] \cap \mathbb{Q}$ is open.

7. The function $f(x) = e^{\frac{1}{x}}$ has a jump discontinuity at $x = 0$.

8. Let $f(x) = (x - 2)\ln(x^2 + 1)$ for $x \in [0, 2]$. Then there exists a point in $(0, 2)$ at which the tangent line is parallel to the x-axis.

9. The function $f(x) = 2^x - \frac{x+2}{x^2+1}$ has a root on the interval $[0, 1]$.

10. There exists a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ that is not bounded.

11. If the function f is differentiable at x_0 from the right and from the left then f is differentiable at x_0.

12. If the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has an inflection point at x_0, then $f''(x_0) = 0$ and $f'''(x_0) \neq 0$.

13. If the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is odd, then f' is even.

14. The partial fraction decomposition of $f(x) = \frac{1}{x^4 - 16}$ cannot contain the term $\frac{1}{(x+2)^2}$.

15. There exists a function $f : [0, 2] \longrightarrow \mathbb{R}$ that is Riemann integrable but doesn’t have an antiderivative.

Answers

I. Definitions and theorems (15 x 3 points)

1. What does it mean that the sequence (a_n) is a Cauchy sequence?

Definition. (a_n) is a Cauchy sequence if for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that
if $n, m > N$ then $|a_n - a_m| < \varepsilon$.

2. Define the limes inferior of the sequence (a_n).

Definition.
- If the set of limit points of (a_n) is bounded below, then its infimum is called the limes inferior of (a_n) (notation: $\liminf a_n$).
- If (a_n) is not bounded below, then we define $\liminf a_n = -\infty$.

3. State the ratio test for number series.

Theorem. Assume that $a_n > 0$. Then

(1) if $\limsup \frac{a_{n+1}}{a_n} < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent;

(2) if $\liminf \frac{a_{n+1}}{a_n} > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

4. State the comparison test for number series.

Theorem. Assume that $0 \le c_n \le a_n \le b_n$ for $n > N$ where N is some fixed integer. Then

(1) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(2) If $\sum_{n=1}^{\infty} c_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

5. What does it mean that the number $x \in \mathbb{R}$ is a limit point of the set $A \subset \mathbb{R}$?

Definition. Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then x is a limit point of A, if for all $r > 0$: $(B(x, r) \setminus \{x\}) \cap A \neq \emptyset$
It means that any interval $(x - r, x + r)$ contains a point in A that is distinct from x.

6. What does it mean that the limit of the function f at $x_0 \in \mathbb{R}$ is $A \in \mathbb{R}$?

Definition. The limit of the function $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ is $A \in \mathbb{R}$ if

(1) x_0 is a limit point of D_f $(x \in D_f')$

(2) for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that
if $x \in D_f$ and $0 < |x - x_0| < \delta(\varepsilon)$ then $|f(x) - A| < \varepsilon$.

7. State the sequential criterion for continuity.

Theorem. The function $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $x_0 \in D_f$ if and only if
for all sequences $(x_n) \subset D_f$ for which $x_n \longrightarrow x_0$, $\lim_{n \to \infty} f(x_n) = f(x_0)$.

8. What does it mean that a function f has an essential discontinuity at $x_0 \in \mathbb{R}$?

Definition. f has an essential discontinuity at x_0, if at least one of the one-sided limits at x_0 doesn't exist or exists but is not finite.

9. State Weierstrass' extreme value theorem for continuous functions.

Theorem. If f is continuous on the closed interval $[a, b]$ then there exist numbers $\alpha \in [a, b]$ and $\beta \in [a, b]$, such that $f(\alpha) \leq f(x) \leq f(\beta)$ for all $x \in [a, b]$,
that is, f has both a minimum and a maximum on $[a, b]$.

10. What does it mean that a function is concave? Write down the definition.

Definition. The function f is concave on the interval $I \subset D_f$ if for all $x, y \in I$ and $t \in [0, 1]$
$f(t x + (1 - t) y) \geq t f(x) + (1 - t) f(y)$

Or:
Definition. Let $h_{a,b}(x)$ denote the the secant line passing through the points $(a, f(a))$ and $(b, f(b))$.
The function f is concave on the interval $I \subset D_f$ if for all $\forall a, b \in I$ and $a < x < b \implies f(x) \geq h_{a,b}(x)$,
that is, the secant lines of f always lie below the graph of f.

11. State the theorem about the derivative of the inverse of a function.

Theorem. Assume that f is continuous and strictly monotonic on (a, b),
f is differentiable at $c \in (a, b)$ and $f'(c) \neq 0$. Then f^{-1} is differentiable at $f(c)$ and

$$\left(f^{-1}\right)'(f(c)) = \frac{1}{f'(c)}$$

12. State the L'Hospital's rule.

Theorem.
Assume that $a \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, I is a neighbourhood of a, the functions f and g are differentiable on $I \setminus \{a\}$ and $g(x) \neq 0$, $g'(x) \neq 0$ for all $x \in I \setminus \{a\}$. Assume moreover that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \quad \text{or} \quad \lim_{x \to a} |f(x)| = \lim_{x \to a} |g(x)| = \infty.$$

If $\exists \lim_{x \to a} \frac{f'(x)}{g'(x)} = b \in \overline{\mathbb{R}}$ then $\exists \lim_{x \to a} \frac{f(x)}{g(x)} = b$.

13. Give two sufficient conditions for a function to have a local minimum at the point x_0.

Theorems.
1) Assume that f is differentiable at $x_0 \in \operatorname{int} D_f$.
If $f'(x_0) = 0$ and f' changes sign from negative to positive at x_0, then f has a local minimum at x_0.
2) Assume that f is twice differentiable at $x_0 \in \operatorname{int} D_f$.
If $f'(x_0) = 0$ and $f''(x_0) > 0$ then f has a local minimum at x_0.

14. State the Newton-Leibniz formula.

Theorem. If $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable and $F : [a, b] \longrightarrow \mathbb{R}$ is an antiderivative of f, that is, $F'(x) = f(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b$.

15. State the second fundamental theorem of calculus.

Theorem. Assume that f is Riemann integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$. Then

1. F is Lipschitz continuous on $[a, b]$.
2. If f is continuous at $x_0 \in [a, b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

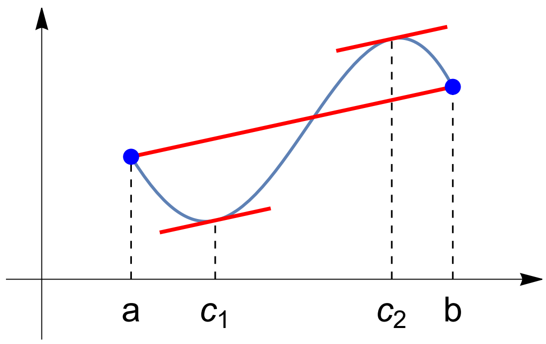
II. Proof of a theorem (15 points)

Write down the statement of Lagrange's mean value theorem and prove it.

Theorem (Lagrange's mean value theorem).

Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b).

Then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



Proof. The equation of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ is

$$y = h_{a,b}(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x) = f(x) - h_{a,b}(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a) - f(a)$.

Then

1) g is continuous on $[a, b]$

2) g is differentiable on (a, b)

3) $g(a) = g(b) = 0$

$\Longrightarrow$ by Rolle's theorem there exists $c \in (a, b)$ such that $g'(c) = 0$

$$\Longrightarrow g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0.$$

III. True or false? (15 x 3 points)

1. If a sequence is monotonic and bounded then it has only one real limit point.

True, since if a sequence is monotonic and bounded then it is convergent.

2. If (a_n) is a nonnegative sequence and $\lim_{n\to\infty} a_n = 1$ then $\lim_{n\to\infty} a_n^n = 1$.

False. For example, $a_n = 1 + \frac{1}{n} \longrightarrow 1$, but $a_n^n = \left(1 + \frac{1}{n}\right)^n = e$.

3. If the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

True. If $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges then by the nth term test $\sqrt{a_n} \longrightarrow 0$. Then by the definition of the limit there exists $n \in \mathbb{N}$ such that for all $n > \mathbb{N}$, $0 \leq \sqrt{a_n} < 1$. From this is follows that $0 \leq a_n \leq \sqrt{a_n} < 1$ also holds. Since $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges, then by the comparison test $\sum_{n=1}^{\infty} a_n$ also converges.

4. If the series $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

False. For example, if $a_n = \frac{1}{n}$, then $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

5. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} \sqrt[n]{|\, a_n \,|} < 1$.

False. For example, if $a_n = \frac{(-1)^n}{n}$ or $a_n = \frac{1}{n^2}$ then $\sum_{n=1}^{\infty} a_n$ is convergent, but $\lim_{n\to\infty} \sqrt[n]{|\, a_n \,|} = 1$.

6. The set $H = [0, 1] \cap \mathbb{Q}$ is open.

False. If $x \in H$ and I is an interval containing x then I contains irrational numbers, so I cannot be a subset of H.

7. The function $f(x) = e^{\frac{1}{x}}$ has a jump discontinuity at $x = 0$.

False. Since $\lim_{x\to 0-0} \frac{1}{x} = -\infty$ and $\lim_{x\to 0+0} \frac{1}{x} = +\infty$, then $\lim_{x\to 0-0} f(x) = 0$ and $\lim_{x\to 0+0} f(x) = \infty$, so f has an essential discontinuity at $x = 0$.

8. Let $f(x) = (x - 2)\ln\left(x^2 + 1\right)$ for $x \in [0, 2]$. Then there exists a point in $(0, 2)$ at which the tangent line is parallel to the x-axis.

True. Since f is differentiable on $[0, 2]$ and $f(0) = f(2)$, then by Rolle's theorem there exists $c \in (0, 2)$, such that $f'(c) = 0$.

9. The function $f(x) = 2^x - \frac{x+2}{x^2+1}$ has a root on the interval $[0, 1]$.

True. $f(0) = -1 < 0$ and $f(1) = \frac{1}{2} > 0$, so by Bolzano's theorem f has a real root on the interval $[0, 1]$.

10. There exists a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ that is not bounded.

False. By Weierstrass boundedness theorem, if f is continuous on $[a, b]$, then f is bounded on $[a, b]$.

11. If the function f is differentiable at x_0 from the right and from the left then f is differentiable at x_0.

False. For example, if $f(x) = |x|$ then $f_+'(0) = \lim_{x \to 0+0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+0} \frac{x - 0}{x - 0} = 1$ and

$f_-'(0) = \lim_{x \to 0-0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-0} \frac{-x - 0}{x - 0} = -1$. Since $f_-'(0) \neq f_+'(0)$, then f is not differentiable at $x_0 = 0$.

12. If the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has an inflection point at x_0, then $f''(x_0) = 0$ and $f'''(x_0) \neq 0$.

False. For example, $f(x) = x^5$ has an inflection point at $x_0 = 0$, but $f''(0) = f'''(0) = 0$.

13. If the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is odd, then f' is even.

True. Since f is odd, then $f(x) = -f(-x)$, so $f'(x) = -f'(-x) \cdot (-1) = f'(-x)$, therefore f' is even.

14. The partial fraction decomposition of $f(x) = \frac{1}{x^4 - 16}$ cannot contain the term $\frac{1}{(x+2)^2}$.

True. The partial fraction decomposition of $f(x) = \frac{1}{x^4 - 16} = \frac{1}{(x^2 - 4)(x^2 + 4)} = \frac{1}{(x-2)(x+2)(x^2+4)}$ is

$\frac{A}{x-2} + \frac{B}{x+2} + \frac{C\,x + D}{x^2 + 4}$

15. There exists a function $f : [0, 2] \longrightarrow \mathbb{R}$ that is Riemann integrable but doesn't have an antiderivative.

True. For example, let $f : [0, 2] \longrightarrow \mathbb{R}$, $f(x) = 1$ if $0 \le x < 1$ and $f(x) = 2$ if $1 \le x \le 2$. Then f is Riemann integrable, since it is bounded and f is discontinuous only at one point, $x = 1$. By Darboux theorem, f doesn't have an antiderivative, since it has a jump discontinuity.