Calculus 1, Final exam 3, Part 2

23rd January, 2023

Name: ____________________________________ Neptun code: _______________

1.: _____ 2.: _____ 3.: _____ 4.: _____ 5.: _____ 6.: _____ 7.: _____ 8.: _____ Sum: _____

1. (10 points) Calculate the following limit: $\lim_{x\to 0} \frac{e^{3x} - 1 - \sin(3x)}{x \ln(2x+1)}$

2. (5+5+5 points) Calculate the derivatives of the following functions:

a) $f:(0,\infty)\longrightarrow\mathbb{R},\ f(x) = (\arctan(2x))^x$

b) $f:(0,\infty)\longrightarrow\mathbb{R},\ f(x) = e^{\sqrt{x}} \cdot \ln\left(\frac{x^2+1}{\cos x + 3}\right)$

c) $f(x) = \begin{cases} x^3 \cdot \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

In case c), use the definition to calculate $f'(0)$.

3. (10 points) The sum of the lengths of the edges of a square prism is 1m. What are the edge lengths of the prism with maximal volume? What is this maximal volume?

4. (15 points) Analyze the following function and sketch its graph: $f(x) = \frac{x}{x^2+3}$.

5. (10+10 points) Calculate the following integrals:

a) $I_1 = \int x \arctan\left(x^2\right) \mathrm{dx}$

b) $I_2 = \int \sin\left(\sqrt{x}\right) \mathrm{dx}$ (substitution: $t = \sqrt{x}$)

6. (10+10 points) Calculate the following integrals:

a) $I_3 = \int \frac{4x+1}{(x-4)\left(x^2+1\right)} \mathrm{dx}$

b) $I_4 = \int \frac{e^x}{e^{2x} + 4e^x - 5} \mathrm{dx}$ (substitution: $t = e^x$)

7. (10 points) Consider the function $f(x) = \frac{x}{\sqrt[4]{x^3+8}}$ on the interval $x \in [1, 2]$. Rotate it around the x-axis and find the volume of the arising body.

8.* (10 points - BONUS)

The windows of a house are designed such that the bottom part is a rectangle and the upper part is a semicircle fitting the rectangle. The perimeter of a window is P. What should be the dimensions of the windows to let in as much light as possible?

Solutions

1. (10 points) Calculate the following limit: $\lim_{x\to 0} \frac{e^{3x} - 1 - \sin(3x)}{x \ln(2x+1)}$

Solution. The limit has the form $\frac{0}{0}$, so the L'Hospital's rule can be applied:

$$\lim_{x\to 0} \frac{e^{3x} - 1 - \sin(3x)}{x \ln(2x+1)} \overset{\frac{0}{0}, L'H}{=} \lim_{x\to 0} \frac{3e^{3x} - 3\cos(3x)}{\ln(2x+1) + \frac{2x}{2x+1}} \textbf{ (4p)} \overset{\frac{0}{0}, L'H}{=} \lim_{x\to 0} \frac{9e^{3x} + 9\sin(3x)}{\frac{2}{2x+1} + \frac{2(2x+1) - 2x\cdot 2}{(2x+1)^2}} \textbf{ (4p)}$$

$$= \frac{9+0}{2+2} = \frac{9}{4} \textbf{ (2p)}$$

2. (5+5+5 points) Calculate the derivatives of the following functions:

a) $f : (0, \infty) \longrightarrow \mathbb{R},\ f(x) = (\arctan(2x))^x$ b) $f : (0, \infty) \longrightarrow \mathbb{R},\ f(x) = e^{\sqrt{x}} \cdot \ln\left(\frac{x^2+1}{\cos x + 3}\right)$

c) $f(x) = \begin{cases} x^3 \cdot \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

In case c), use the definition to calculate $f'(0)$.

a) $f(x) = (\arctan(2x))^x = e^{\ln((\arctan(2x))^x)} = e^{x \ln(\arctan(2x))}$

$\Longrightarrow f'(x) = e^{x \ln(\arctan(2x))} \cdot (x \ln(\arctan(2x)))' =$

$$= (\arctan(2x))^x \cdot \left(\ln(\arctan(2x)) + x \cdot \frac{1}{\arctan(2x)} \cdot \frac{1}{1+(2x)^2} \cdot 2\right) \textbf{ (5p)}$$

b) $f(x) = e^{\sqrt{x}} \cdot \ln\left(\frac{x^2+1}{\cos x + 3}\right)$

$$\Longrightarrow f'(x) = \frac{1}{2\sqrt{x}} e^{\sqrt{x}} \cdot \ln\left(\frac{x^2+1}{\cos x + 3}\right) + e^{\sqrt{x}} \cdot \frac{\cos x + 3}{x^2+1} \cdot \frac{2x(\cos x + 3) - (x^2+1)(-\sin x)}{(\cos x + 3)^2} \textbf{ (5p)}$$

c) If $x \neq 0$ then $f'(x) = 3x^2 \cos\left(\frac{1}{x}\right) + x^3 \cdot \left(-\sin\left(\frac{1}{x}\right)\right) \cdot \frac{-1}{x^2}$ **(2p)**

If $x = 0$ then by the definition of the derivative

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^3 \cdot \cos\left(\frac{1}{x}\right) - 0}{x - 0} = \lim_{x\to 0} x^2 \cdot \cos\left(\frac{1}{x}\right) = 0, \text{ since}$$

the first term of the product tends to zero, and the second term is bounded. **(3p)**

3. (10 points) The sum of the lengths of the edges of a square prism is 1m. What are the edge lengths of the prism with maximal volume? What is this maximal volume?

Solution. Let x denote the side of the square base and let y denote the height of the prism.

Then the volume of the prism is $V = x^2 y$ and the sum of the lengths of the edges is $8x + 4y = 1$.

From here $y = \frac{1 - 8x}{4}$. Substituting this into the volume, we get that

$$V(x) = x^2 \cdot \frac{1 - 8x}{4} = \frac{1}{4}(x^2 - 8x^3).$$

We want to find the maximum of this function if $0 < x < \frac{1}{8}$. **(3p)**

$V'(x) = \frac{1}{4}\left(2x - 24x^2\right) = \frac{1}{4} \cdot 2x(1 - 12x) = 0 \iff x_1 = 0, \quad x_2 = \frac{1}{12}$ **(3p)**

Because of the conditions, $x = 0$ cannot be the case.

$V''(x) = \frac{1}{4}(2 - 48x)$ and $V''\left(\frac{1}{12}\right) = \frac{1}{4}\left(2 - 48 \cdot \frac{1}{12}\right) = -\frac{1}{2} < 0$, so V has a maximum at $x = \frac{1}{12}$. **(2p)**

The sides of the prism with maximal volume are $\frac{1}{12}$, $\frac{1}{12}$ and $\frac{1}{12}$ m (so it is a cube) and the maximum of the

volume is $\frac{1}{12^3}$ m^3. **(2p)**

4. (15 points) Analyze the following function and sketch its graph: $f(x) = \frac{x}{x^2 + 3}$.

Solution.

1) The domain of f is $D_f = \mathbb{R}$.

The zeros of f are: $\frac{x}{x^2 + 3} = 0 \implies x = 0$

The limits of f at $\pm\infty$ are:

$\lim_{x \to \infty} f(x) = 0, \quad \lim_{x \to -\infty} f(x) = 0.$ **(2p)**

2) $f'(x) = \frac{\left(x^2 + 3\right) - x \cdot 2x}{\left(x^2 + 3\right)^2} = \frac{-x^2 + 3}{\left(x^2 + 3\right)^2} = 0 \implies x_{1,2} = \pm\sqrt{3}$ **(2p)**

x	$x<-\sqrt{3}$	$x=-\sqrt{3}$	$-\sqrt{3}<x<\sqrt{3}$	$x=\sqrt{3}$	$x>\sqrt{3}$
f'	–	0	+	0	–
f	↘	min: $-\frac{1}{2\sqrt{3}}$	↗	max: $\frac{1}{2\sqrt{3}}$	↘

(3p)

3) $f''(x) = \frac{-2x\left(x^2 + 3\right)^2 - \left(-x^2 + 3\right) \cdot 2 \cdot \left(x^2 + 3\right) \cdot 2x}{\left(x^2 + 3\right)^4} = \frac{-2x\left(x^2 + 3\right) - \left(-x^2 + 3\right) \cdot 2 \cdot 2x}{\left(x^2 + 3\right)^3} =$

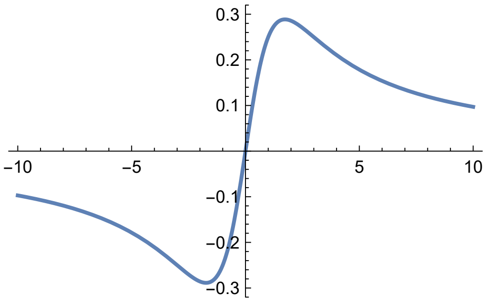
$= \frac{-2x^3 - 6x + 4x^3 - 12x}{\left(x^2 + 3\right)^2} = \frac{2x^3 - 18x}{\left(x^2 + 3\right)^2} = \frac{2x\left(x^2 - 9\right)}{\left(x^2 + 3\right)^2} = \frac{2x(x - 3)(x + 3)}{\left(x^2 + 3\right)^2} = 0$

$\implies x_1 = 2, \quad x_{2,3} = \pm 3$ **(2p)**

x	$x<-3$	$x=-3$	$-3<x<0$	$x=0$	$0<x<3$	$x=3$	$x>3$
f''	–	0	+	0	–	0	+
f	∩	infl: $-\frac{1}{4}$	∪	infl: 0	∩	infl: $\frac{1}{4}$	∪

(3p)

The graph of f: **(3p)**



$\Longrightarrow$ the range of f is $\left[-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right]$

5. (10+10 points) Calculate the following integrals:

a) $I_1 = \int x \arctan(x^2)\, dx$ b) $I_2 = \int \sin(\sqrt{x})\, dx$ (substitution: $t = \sqrt{x}$)

Solution: a) We use the integration by parts method: $\int f' \cdot g = f \cdot g - \int f \cdot g'$

- $f'(x) = x \quad \Longrightarrow f(x) = \frac{x^2}{2}$
- $g(x) = \arctan(x^2) \quad \Longrightarrow g'(x) = \frac{1}{1+x^4} \cdot 2x$

$$\Longrightarrow I_1 = \int x \arctan(x^2)\, dx = \frac{x^2}{2} \cdot \arctan(x^2) - \int \frac{x^2}{2} \cdot \frac{1}{1+x^4} \cdot 2x\, dx =$$

$$= \frac{x^2}{2} \cdot \arctan(x^2) - \int \frac{1}{4} \cdot \frac{4x^3}{1+x^4}\, dx = \frac{x^2}{2} \cdot \arctan(x^2) - \frac{1}{4} \ln(1+x^4) + c$$

b) $I_2 = \int \sin(\sqrt{x})\, dx = ?$ Substitution: $t = \sqrt{x} \Longrightarrow x = x(t) = t^2 \Longrightarrow x'(t) = \frac{dx}{dt} = 2t \Longrightarrow dx = 2t\, dt$

$$\Longrightarrow I_2 = \int 2t \cdot \sin(t)\, dt \quad \textbf{(5p)}$$

We use the integration by parts method: $\int f' \cdot g = f \cdot g - \int f \cdot g'$

- $f'(t) = \sin t \quad \Longrightarrow f(t) = -\cos t$
- $g(t) = 2t \quad \Longrightarrow g'(x) = 2$

$$\Longrightarrow I_2 = -2t \cos t - \int 2 \cdot (-\cos t)\, dt = -2t \cos t + 2 \sin t + c = -2\sqrt{x} \cos(\sqrt{x}) + 2 \sin(\sqrt{x}) + c = \textbf{(5p)}$$

6. (10+10 points) Calculate the following integrals:

a) $I_3 = \int \frac{4x+1}{(x-4)(x^2+1)}\, dx$ b) $I_4 = \int \frac{e^x}{e^{2x} + 4e^x - 5}\, dx$ (substitution: $t = e^x$)

Solution. a) We use partial fraction decomposition:

$\frac{4x+1}{(x-4)(x^2+1)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+1}$ **(2p)** Multiplying by $(x-4)(x^2+1)$ we get:

$$4x+1 = A(x^2+1) + (x-4)(Bx+C)$$

$x = 4 \Longrightarrow \quad 17 = 17A + 0 \quad \Longrightarrow A = 1$

$x = 0 \Longrightarrow \quad 1 = A - 4C \quad \Longrightarrow C = 0$ **(3p)**

$x = 1 \Longrightarrow \quad 5 = 2A - 3(B+C) \quad \Longrightarrow 3B = 2A - 5 \Longrightarrow B = -1$

$$\Longrightarrow I_3 = \int \frac{4x+1}{(x-4)(x^2+1)}\,\mathrm{dx} = \int\left(\frac{1}{x-4} - \frac{x}{x^2+1}\right)\mathrm{dx} = \int\left(\frac{1}{x-4} - \frac{1}{2}\,\frac{2x}{x^2+1}\right)\mathrm{dx} =$$

$$= \ln\left|\,x-4\,\right| - \frac{1}{2}\ln(x^2+1) + c$$ **(5p)**

b) $I_4 = \int \frac{e^x}{e^{2x} + 4e^x - 5}\,\mathrm{dx} = ?$ (substitution: $t = e^x$)

Substitution: $t = e^x \Longrightarrow x = x(t) = \ln t \Longrightarrow x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{1}{t} \Longrightarrow \mathrm{dx} = \frac{1}{t}\,\mathrm{dt}$

$$\Longrightarrow I_4 = \int \frac{e^x}{e^{2x} + 4e^x - 5}\,\mathrm{dx} = \int \frac{t}{t^2 + 4t - 5}\cdot\frac{1}{t}\,\mathrm{dt} = \int \frac{1}{(t-1)(t+5)}\,\mathrm{dt}$$ **(4p)**

Partial fraction decomposition: $\frac{1}{(t-1)(t+5)} = \frac{A}{t-1} + \frac{B}{t+5}$

$\Longrightarrow \quad 1 = A(t+5) + B(t-1)$

$t = 1 \quad \Longrightarrow \quad 1 = 6A + 0 \quad \Longrightarrow A = \frac{1}{6}$

$t = -5 \quad \Longrightarrow \quad 1 = 0 - 6B \quad \Longrightarrow B = -\frac{1}{6}$ **(3p)**

$$\Longrightarrow I_4 = \int\left(\frac{1}{6}\,\frac{1}{t-1} - \frac{1}{6}\,\frac{1}{t+5}\right)\mathrm{dt} = \frac{1}{6}\ln\left|\,t-1\,\right| - \frac{1}{6}\ln\left|\,t+5\,\right| + c = \frac{1}{6}\ln\left|\,e^x - 1\,\right| - \frac{1}{6}\ln(e^x+5) + c$$ **(3p)**

7. (10 points) Consider the function $f(x) = \frac{x}{\sqrt[4]{x^3+8}}$ on the interval $x \in [1, 2]$. Rotate it around the x-axis and find the volume of the arising body.

Solution. The volume is $V = \pi\int_0^{\pi} f^2(x)\,\mathrm{dx} = \pi\int_1^2 \frac{x^2}{\sqrt{x^3+8}}\,\mathrm{dx}$ **(2p)**

$$= \pi\int_1^2 x^2\cdot(x^3+8)^{-\frac{1}{2}}\,\mathrm{dx} = \pi\int_1^2 \frac{1}{3}\cdot 3x^2\cdot(x^3+8)^{-\frac{1}{2}}\,\mathrm{dx}$$ **(3p)**

$$= \frac{\pi}{3}\left[\frac{(x^3+8)^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^2$$ **(2p)** $$= \frac{2\pi}{3}\left[\sqrt{x^3+8}\right]_1^2 = \frac{2\pi}{3}\left(\sqrt{16} - \sqrt{9}\right) = \frac{2\pi}{3}$$ **(3p)**

8.* (10 points - BONUS)
The windows of a house are designed such that the bottom part is a rectangle and the upper part is a semicircle fitting the rectangle. The perimeter of a window is P. What should be the dimensions of the windows to let in as much light as possible?

Solution. Let r denote the radius of the semicircle, then the horizontal side of the rectangle is $2r$. Let x denote the vertical side of the rectangle.

The perimeter of the window is $P = 2x + 2r + r\pi$ and its area is $A = \frac{1}{2}r^2\pi + 2rx$.

From the perimeter $2x = P - 2r - r\pi$. Substituting into the area we get

$A(r) = \frac{1}{2}r^2\pi + r(P - 2r - r\pi) = Pr - 2r^2 - \frac{1}{2}r^2\pi$.

We want to find the maximum of this function.

$A'(r) = P - 4r - r\pi = 0 \implies r = \frac{P}{4+\pi}$.

$A''(r) = -4 - \pi < 0$, so $A(r)$ has a maximum for $r = \frac{P}{4+\pi}$.