Calculus 1 - Homework 2.

1. (3+3 points) Calculate the limit of the following sequences:

a) $a_n = \left(\dfrac{3n+4}{3n+7}\right)^{2n}$ **b)** $a_n = \left(\dfrac{n^2+2n}{n^2+2}\right)^{n^2}$

2. (3+3 points) Calculate the limit of the following sequences it if exists:

a) $a_n = \sqrt[n]{\dfrac{n^3-4n^2+8}{n^4+3n^3-7n}}$ **b)** $a_n = \sqrt[n]{3^n + 5^{(-1)^n \cdot n}}$

3. (5 points) Let $a_1 = 3$ and $a_{n+1} = \sqrt[3]{5a_n^2 - 4a_n}$ for all $n \in \mathbb{N}$. Investigate the convergence of (a_n).

4. (3 points) Evaluate the sum of the following series: $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n+1} + (-5)^{n-1}}{3^{2n+1}}$

5. (3+3 points) Decide whether the following series are convergent or divergent:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 + \sqrt{n} + 2}{2n^6 - n^4 + 3n}$ **b)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n^{10} + 6n + 1}}$

6.* (4 points) Calculate the limit of the following sequence if it exists:

$x_n = \sqrt[n^2]{(2n+1)! - (2n)!}$

Deadline: October 18th

Solutions

1. (3+3 points) Calculate the limit of the following sequences:

a) $a_n = \left(\dfrac{3n+4}{3n+7}\right)^{2n}$ **b)** $a_n = \left(\dfrac{n^2+2n}{n^2+2}\right)^{n^2}$

Solution.

a) $$a_n = \left(\frac{1+\frac{4}{3n}}{1+\frac{7}{3n}}\right)^{2n} = \frac{\left(\left(1+\frac{4}{3n}\right)^n\right)^2}{\left(\left(1+\frac{7}{3n}\right)^n\right)^2} \longrightarrow \frac{\left(e^{\frac{4}{3}}\right)^2}{\left(e^{\frac{7}{3}}\right)^2} = e^{\frac{8}{3}-\frac{14}{3}} = e^{-2}$$

b) $a_n = \dfrac{\left(1+\frac{2}{n}\right)^{n^2}}{\left(1+\frac{2}{n^2}\right)^{n^2}} \geq \dfrac{\left(\left(1+\frac{2}{n}\right)^n\right)^n}{e^2} \geq \dfrac{3^n}{e^2} \longrightarrow \infty$, therefore $a_n \longrightarrow \infty$.

We use that since $\left(1+\dfrac{2}{n}\right)^n \longrightarrow e^2$, then the terms are greater then 3, if n is large enough.

2. (3+3 points) Calculate the limit of the following sequences it if exists:

a) $a_n = \sqrt[n]{\dfrac{n^3-4n^2+8}{n^4+3n^3-7n}}$ **b)** $a_n = \sqrt[n]{3^n+5^{(-1)^n \cdot n}}$

Solution. a) An upper estimation:

$$a_n = \sqrt[n]{\frac{n^3-4n^2+8}{n^4+3n^3-7n}} \leq \sqrt[n]{\frac{n^3+0+8n^3}{n^4+0-\frac{1}{2}n^4}} = \sqrt[n]{\frac{9n^3}{\frac{1}{2}n^4}} = \sqrt[n]{\frac{18}{n}} = \frac{\sqrt[n]{18}}{\sqrt[n]{n}} \longrightarrow \frac{1}{1} = 1$$

The first estimation is true if $\frac{1}{2}n^4 \geq 7n$, that is, $n \geq 3$.

A lower estimation:

$$a_n = \sqrt[n]{\frac{n^3-4n^2+8}{n^4+3n^3-7n}} \geq \sqrt[n]{\frac{n^3-\frac{1}{2}n^3+0}{n^4+3n^4+0}} \geq \sqrt[n]{\frac{\frac{1}{2}n^3}{4n^4}} = \sqrt[n]{\frac{1}{8n}} = \frac{1}{\sqrt[n]{8}\cdot\sqrt[n]{n}} \longrightarrow \frac{1}{1\cdot 1} = 1$$

The first estimation is true if $\frac{1}{2}n^3 \geq 4n^2$, that is, $n \geq 8$.

So by the sandwich theorem $a_n \longrightarrow 1$.

b) If n is even, then $a_n = \sqrt[n]{3^n+5^n}$, so

$5 = \sqrt[n]{0+5^n} \leq a_n \leq \sqrt[n]{5^n+5^n} = \sqrt[n]{2\cdot 5^n} = \sqrt[n]{2}\cdot 5 \longrightarrow 1\cdot 5 = 5$

If n is odd, then $a_n = \sqrt[n]{3^n+\left(\dfrac{1}{5}\right)^n}$, so

$3 = \sqrt[n]{3^n+0} \leq a_n \leq \sqrt[n]{3^n+3^n} = \sqrt[n]{2\cdot 3^n} = \sqrt[n]{2}\cdot 3 \longrightarrow 1\cdot 3 = 3$

Therefore, $\liminf a_n = 3$ and $\limsup a_n = 5$ so the limit of a_n doesn't exist.

3. (5 points) Let $a_1 = 3$ and $a_{n+1} = \sqrt[3]{5a_n^2 - 4a_n}$ for all $n \in \mathbb{N}$. Investigate the convergence of (a_n).

Solution. Let $A = \lim\limits_{n\to\infty} a_n$, then the solutions of the equation $A = \sqrt[3]{5A^2-4A}$ are $A_1 = 0,\ A_2 = 1, A_3 = 4$.

(1) By induction we show that $1 \leq a_n \leq 4$ for all $n \in \mathbb{N}$.

If $n = 1$ then $1 \leq a_1 = 3 \leq 4$ holds.

Assume that $1 \leq a_n \leq 4$. Then

$1 \leq a_n \leq 4 \Longrightarrow 1 \leq 5a_n - 4 \leq 16$

$\Longrightarrow 1 \leq a_n \cdot (5a_n - 4) = 5a_n^2 - 4a_n \leq 64$

$\Longrightarrow 1 \leq \sqrt[3]{5a_n^2-4a_n} = a_{n+1} \leq 4.$

(2) We show that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

$a_n \leq a_{n+1} \iff a_n^3 \leq 5a_n^2 - 4a_n \iff$

$a_n^3 - 5a_n^2 + 4a_n = a_n(a_n^2 - 5a_n + 4) = a_n(a_n-1)(a_n-4) \leq 0.$

This holds for all $n \in \mathbb{N}$, since $a_n > 0$ and $1 \leq a_n \leq 4$.

We showed that (a_n) is monotonically increasing and bounded, so it is convergent.
Since (a_n) is monotonically increasing, then $A \geq a_1 = 3$, so $A_1 = 0$, and $A_2 = 1$ cannot be the limit, and thus $\lim_{n\to\infty} a_n = 4$.

4. (3 points) Evaluate the sum of the following series: $\sum_{n=1}^{\infty} \frac{2^{3n+1} + (-5)^{n-1}}{3^{2n+1}}$

Solution. $\sum_{n=1}^{\infty} \frac{2^{3n+1} + (-5)^{n-1}}{3^{2n+1}} = \sum_{n=1}^{\infty}\left(\frac{2}{3}\cdot\left(\frac{8}{9}\right)^n - \frac{1}{15}\cdot\left(-\frac{5}{9}\right)^n\right) = \frac{2}{3}\cdot\frac{\frac{8}{9}}{1-\frac{8}{9}} - \frac{1}{15}\cdot\frac{-\frac{5}{9}}{1+\frac{5}{9}} = \frac{75}{14}$

5. (3+3 points) Decide whether the following series are convergent or divergent:

a) $\sum_{n=1}^{\infty} \frac{3n^2 + \sqrt{n} + 2}{2n^6 - n^4 + 3n}$ **b)** $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n^{10} + 6n + 1}}$

Solution.

a) $0 < a_n = \frac{3n^2 + \sqrt{n} + 2}{2n^6 - n^4 + 3n} \leq \frac{3n^2 + n^2 + 2n^2}{2n^6 - n^6 + 0} = \frac{6}{n^4}$ and $\sum_{n=1}^{\infty} \frac{6}{n^4}$ is convergent, so by the comparison test the series $\sum_{n=1}^{\infty} a_n$ is also convergent.

b) $1 \leq b_n = \sqrt[n]{n^{10} + 6n + 1} \leq \sqrt[n]{n^{10} + 6n^{10} + n^{10}} = \sqrt[n]{8n^{10}} = \sqrt[n]{8}\cdot\left(\sqrt[n]{n}\right)^{10} \longrightarrow 1$,

so by the sandwich theorem $b_n \longrightarrow 1$. Then $a_n = \frac{1}{b_n} = \frac{1}{\sqrt[n]{n^{10} + 6n + 1}} \longrightarrow 1$.

Since $\lim_{n\to\infty} a_n \neq 0$ then by the nth term test the series $\sum_{n=1}^{\infty} a_n$ is divergent.

6.* (4 points) Calculate the limit of the following sequence if it exists:

$x_n = \sqrt[n^2]{(2n+1)! - (2n)!}$

Solution. We use the following:

(a) $(2n+1)! - (2n)! = (2n)!\cdot(2n+1-1) = (2n)!\cdot 2n$

(b) $(2n)! = 1\cdot 2\cdot 3\cdot \ldots \cdot(2n) \leq (2n)\cdot(2n)\cdot(2n)\cdot\ldots\cdot(2n) = (2n)^{2n}$

$\Longrightarrow$ $\mathbf{(2n)! \leq (2n)^{2n}}$

(c) Since $\sqrt[n]{2} \longrightarrow 1$ and $\sqrt[n]{n} \longrightarrow 1$ then $\mathbf{\sqrt[n]{2} \leq 2}$ and $\mathbf{\sqrt[n]{n} \leq 2}$, if n is large enough.

$1 \leq x_n = \sqrt[n^2]{(2n+1)! - (2n)!} = \sqrt[n^2]{2n\cdot\mathbf{(2n)!}} \leq \sqrt[n^2]{2n\cdot\mathbf{(2n)^{2n}}} =$

$= \sqrt[n]{\sqrt[n]{2}\cdot\sqrt[n]{n}}\cdot\sqrt[n]{\sqrt[n]{(2n)^{2n}}} \leq \sqrt[n]{2\cdot 2}\cdot\sqrt[n]{4n^2} = \sqrt[n]{4}\cdot\sqrt[n]{4}\cdot\left(\sqrt[n]{n}\right)^2 \longrightarrow 1\cdot 1\cdot 1^2 = 1$

Therefore, because of the sandwich theorem, $x_n \longrightarrow 1$.