Calculus 1 - Homework 3

1. (4 points) Let $A = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup (\mathbb{Q} \cap [1,\ 2]) \cup (3,\ 4]$.

Find the set of interior points, boundary points, limit points and isolated points of A.

Solution.

Set of interior points: $\operatorname{int} A = (3,\ 4)$

Set of boundary points: $\partial A = \left\{-\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\} \cup [1,\ 2] \cup \{3,\ 4\}$

Set of limit points: $A' = \{0\} \cup [1,\ 2] \cup [3,\ 4]$

Set of isolated points: $\left\{-\frac{1}{n} : n \in \mathbb{N}\right\}$

2. (3+3 points) Calculate the following limits:

a) $\lim\limits_{x\to 1} \frac{x^2-1}{\sqrt{x}-\sqrt{2-x}}$ **b)** $\lim\limits_{x\to 0} \frac{\sin^2(a\,x)}{\cos(b\,x)-1}$, where $a,\ b \in \mathbb{R} \setminus \{0\}$.

Solutions.

a) $$\lim_{x\to 1} \frac{x^2-1}{\sqrt{x}-\sqrt{2-x}} = \lim_{x\to 1} \frac{x^2-1}{\sqrt{x}-\sqrt{2-x}} \cdot \frac{\sqrt{x}+\sqrt{2-x}}{\sqrt{x}+\sqrt{2-x}} =$$

$$= \lim_{x\to 1} \frac{(x^2-1)\left(\sqrt{x}+\sqrt{2-x}\right)}{x-(2-x)} = \lim_{x\to 1} \frac{(x-1)(x+1)\left(\sqrt{x}+\sqrt{2-x}\right)}{2(x-1)} =$$

$$= \lim_{x\to 1} \frac{(x+1)\left(\sqrt{x}+\sqrt{2-x}\right)}{2} = \frac{(1+1)(1+1)}{2} = 2$$

b) $$\lim_{x\to 0} \frac{\sin^2(a\,x)}{\cos(b\,x)-1} = \lim_{x\to 0} \frac{\sin^2(a\,x)}{\cos(b\,x)-1} \cdot \frac{\cos(b\,x)+1}{\cos(b\,x)+1} = \lim_{x\to 0} \frac{\sin^2(a\,x)}{\cos^2(b\,x)-1} \cdot (\cos(b\,x)+1) =$$

$$= \lim_{x\to 0} \frac{\sin^2(a\,x)}{-\sin^2(b\,x)} \cdot (\cos(b\,x)+1) = \lim_{x\to 0} \left(\frac{\sin(a\,x)}{a\,x}\right)^2 \cdot \left(\frac{b\,x}{\sin(b\,x)}\right)^2 \cdot \frac{-a^2}{b^2} (\cos(b\,x)+1) =$$

$$= 1^2 \cdot 1^2 \cdot \frac{-a^2}{b^2} \cdot (1+1) = -\frac{2\,a^2}{b^2}$$

3. (4 points) Choose the values of the parameters $a,\ b \in \mathbb{R}$ so that the following function be continuous on $\mathbb{R}$:

$$f(x) = \begin{cases} \frac{\cos^2 x - a}{x} & \text{if } x < 0 \\ \sin^2 \frac{\pi(x+b)}{2} & \text{if } x \geq 0 \end{cases}$$

Solution. f is continuous if $x \neq 0$ for all $a,\ b \in \mathbb{R}$.

At $x = 0$ the function f will be continuous if and only if $\lim\limits_{x\to 0-0} f(x) = \lim\limits_{x\to 0+0} f(x) = f(0)$

(1) $\dfrac{\cos^2 x - a}{x} = \dfrac{(\cos^2 x - 1) + (1 - a)}{x} = \dfrac{(\cos^2 x - 1)}{x} + \dfrac{1 - a}{x} = \dfrac{-\sin^2 x}{x} + \dfrac{1 - a}{x}$

- $\lim\limits_{x \to 0} \dfrac{-\sin^2 x}{x} = \lim\limits_{x \to 0} \dfrac{\sin x}{x} (-\sin x) = 1 \cdot 0 = 0$
- $\dfrac{1 - a}{x} = 0$, if $a = 1$ and $\lim\limits_{x \to 0 \pm 0} \dfrac{1 - a}{x} = \pm\infty$, if $a \neq 1$

$\Longrightarrow$ f has a finite limit at 0 from the left if and only if $a = 1$ and then $\lim\limits_{x \to 0-0} f(x) = 0$

(2) $\lim\limits_{x \to 0+0} f(x) = \lim\limits_{x \to 0+0} \sin^2 \dfrac{\pi(x + b)}{2} = \sin^2 \dfrac{\pi b}{2}$

f is continuous at $x = 0 \iff \sin^2 \dfrac{\pi b}{2} = 0 \iff \dfrac{\pi b}{2} = k\pi$ $(k \in \mathbb{Z}) \iff b = 2k$, where $k \in \mathbb{Z}$.

Therefore f is continuous on $\mathbb{R}$ if and only if $a = 1$ and $b = 2k$, where $k \in \mathbb{Z}$.

4. (3 points) Are the following statements true or false? Give a reason for your answer.
a) There exists a continuous function $f : (-1, 1) \longrightarrow \mathbb{R}$ whose range is $[0, 1]$.
b) There exists a continuous function $f : [-1, 1] \longrightarrow \mathbb{R}$ whose range is $(0, 1)$.
c) There exists a continuous function $f : [-1, 1] \longrightarrow \mathbb{R}$ whose range is $[1, 2] \cup [4, 5]$.

Solution.

a) True. For example: $f(x) = \begin{cases} 0, & \text{if } -1 < x \leq 0 \\ 2x, & \text{if } 0 < x \leq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < x < 1 \end{cases}$

b) False. It follows from the intermediate value theorem and the extreme value theorem that if f is continuous on $[-1, 1]$, then the range of f is a closed and bounded interval.
c) False. By the previous two theorems, the range of f must be a closed and bounded interval.

5. (5 points) Determine the points of discontinuities of the following functions.
What type of discontinuities are these?
a) $f(x) = e^{-\frac{1}{x^2}}$ b) $g(x) = \dfrac{1}{1 - e^x}$ c) $h(x) = \dfrac{1}{1 - e^{\frac{1}{x}}}$

Solution.

a) $\lim\limits_{x \to 0+0} e^{-\frac{1}{x^2}} = \lim\limits_{x \to 0-0} e^{-\frac{1}{x^2}} = e^{-\infty} = 0$

$\Longrightarrow$ f has a removable discontinuity at $x = 0$.

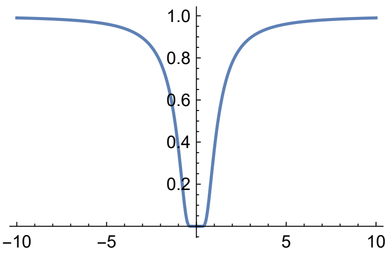
b) $\lim\limits_{x \to 0+0} \dfrac{1}{1 - e^x} = \dfrac{1}{0-} = -\infty$, $\lim\limits_{x \to 0-0} \dfrac{1}{1 - e^x} = \dfrac{1}{0+} = +\infty$

$\Longrightarrow$ f has an essential discontinuity at $x = 0$

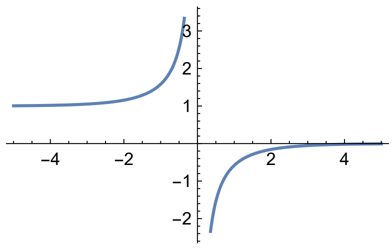
c) $\lim\limits_{x \to 0+0} \dfrac{1}{1 - e^{\frac{1}{x}}} = \dfrac{1}{1 - e^{\infty}} = \dfrac{1}{-\infty} = 0$, $\lim\limits_{x \to 0-0} \dfrac{1}{1 - e^{\frac{1}{x}}} = \dfrac{1}{1 - e^{-\infty}} = \dfrac{1}{1 - 0} = 1$

$\Longrightarrow$ f has a jump continuity at $x = 0$

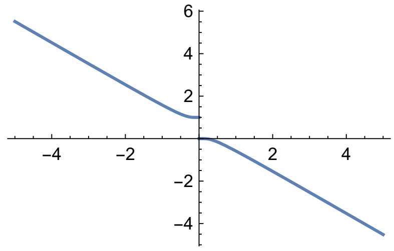
a)



b)



c)



6. (3 points) Let $f(x) = e^{-x}\cos(\pi x) + x^3 - 4$. Prove that f has a zero in the open interval $(0, 2)$.

Solution.
$f(0) = 1 + 0 - 4 = -3 < 0$ and $f(2) = e^{-2} + 8 - 4 > 0$, so by the intermediate value theorem (or Bolzano's theorem) there exists $c \in (0, 2)$ such that $f(c) = 0$.

7.* (4 points) Prove that if f is continuous on $[a, \infty)$ and $\exists \lim_{x\to\infty} f(x) = A \in \mathbb{R}$ then f is uniformly continuous on $[a, \infty)$.

Solution. Let $\varepsilon > 0$ be fixed. Since $\exists \lim_{x\to\infty} f(x) = A \in \mathbb{R}$ then there exists $P > 0$ such that if $x > P$ then $\left| f(x) - A \right| < \frac{\varepsilon}{2}$.

f is continuous, so it is uniformly continuous on the compact interval $[a, P+1]$.
Let $0 < \delta < 1$ such that if $x, y \in [a, P+1]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

Now let $x, y \in [a, \infty)$ such that $|x - y| < \delta$. Then either $x, y \in [a, P+1]$ or $x, y > P$.
($x \le P$, $y > P + 1$ is not possible since their distance is less than 1.)
If $x, y \in [a, P+1]$ then $|f(x) - f(y)| < \varepsilon$.
If $x, y > P$ then $\left| f(x) - f(y) \right| \le \left| f(x) - A \right| + \left| A - f(y) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.