Calculus 1 - Homework 4.

1. (5 points) Find the local extrema of the function $f(x) = \left(x^3 + 3x^2 + 3x - 3\right) e^x$.
Determine the intervals where the function increases or decreases.

Solution. $f'(x) = e^x x(x+3)^2 = 0 \iff x_1 = 0$ and $x_2 = -3$

x	x<−3	x=−3	−3<x<0	x=0	x>0
f'	−	0	−	0	+
f	↘		↘	min:0	↗

f is strictly monotonically decreasing on $(-\infty, 0]$ and strictly monotonically increasing on $[0, \infty)$.

2. (5 points) Find the inflection points of the function $f(x) = (x+1)\,\mathrm{arctg}(x-1)$.
Determine the intervals where the function is convex or concave.

Solution.

$$f'(x) = \mathrm{arctg}(x-1) + \frac{x+1}{1+(x-1)^2}$$

$$f''(x) = \frac{1}{1+(x-1)^2} + \frac{1+(x-1)^2 - (x+1)\cdot 2(x-1)}{\left(1+(x-1)^2\right)^2}$$

$$= \frac{1+(x-1)^2 + 1 + (x-1)^2 - (x+1)\cdot 2(x-1)}{\left(1+(x-1)^2\right)^2} = \frac{2+2x^2-4x+2-2\left(x^2-1\right)}{\left(1+(x-1)^2\right)^2} =$$

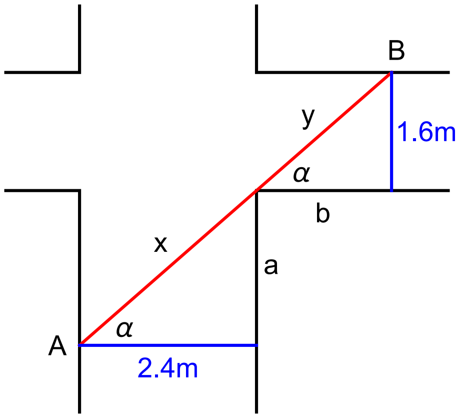
$$= \frac{6-4x}{\left(1+(x-1)^2\right)^2} = 0 \iff x = \frac{3}{2}$$

x	$x<\frac{3}{2}$	$x=\frac{3}{2}$	$x>\frac{3}{2}$
f''	+	0	−
f	⋃	infl.	⋂

f is convex on $\left(-\infty, \frac{3}{2}\right)$ and concave on $\left(\frac{3}{2}, \infty\right) \Longrightarrow f$ has an inflection point at $\frac{3}{2}$.

3.* (4 points) The widths of two perpendicular corridors are 2.4 m and 1.6 m, respectively. What is the longest ladder that can be moved (in a horizontal position) from one corridor to another?

1st solution.



The ladder is denoted by the line segment AB in the figure. The lengths x and y can be expressed with α, so the length of the ladder is $AB = f(\alpha) = \dfrac{2.4}{\cos\alpha} + \dfrac{1.6}{\sin\alpha} \Longrightarrow$

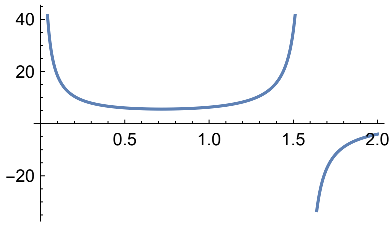
$$f'(\alpha) = \frac{2.4\sin\alpha}{\cos^2\alpha} - \frac{1.6\cos\alpha}{\sin^2\alpha} = \frac{0.8}{\sin^2\alpha\cos^2\alpha}\left(3\sin^3\alpha - 2\cos^3\alpha\right)$$

$$f'(\alpha) = 0 \text{ if } 3\sin^3\alpha = 2\cos^3\alpha \Longrightarrow \tan\alpha = \sqrt[3]{\frac{2}{3}} \Longrightarrow \alpha = \arctan\left(\sqrt[3]{\frac{2}{3}}\right) \approx 0.718025$$

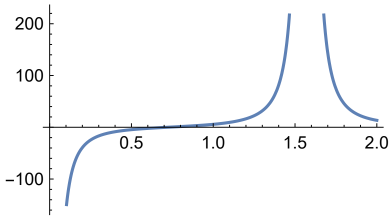
$\Longrightarrow \alpha \approx 41\,°$. The length of the ladder is $AB \approx 3.2 + 2.4 = 5.6$ m

Remark. f has a local minimum at $\alpha \approx 0.718$. f' changes sign from negative to positive.

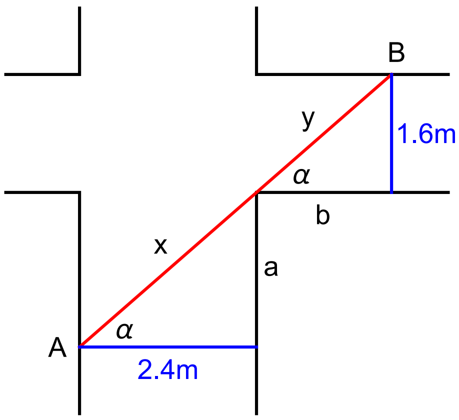
The graph of f:



The graph of f':



2nd solution.



According to the figure, $\dfrac{a}{2.4} = \dfrac{1.6}{b}$, so $a = \dfrac{1.6 \cdot 2.4}{b}$. The length of the ladder is

$$f(b) = \sqrt{2.4^2 + a^2} + \sqrt{1.6^2 + b^2} = \sqrt{2.4^2 + \frac{1.6^2 \times 2.4^2}{b^2}} + \sqrt{1.6^2 + b^2} =$$

$$= 2.4\sqrt{1+\frac{1.6^2}{b^2}} + \sqrt{1.6^2+b^2} = \frac{2.4}{b}\sqrt{1.6^2+b^2} + \sqrt{1.6^2+b^2} = \left(\frac{2.4}{b}+1\right)\sqrt{1.6^2+b^2}$$

$$f'(b) = -\frac{2.4}{b^2}\sqrt{1.6^2+b^2} + \left(\frac{2.4}{b}+1\right)\frac{b}{\sqrt{1.6^2+b^2}} = 0$$

$$\Longleftrightarrow\ 1.6^2+b^2 = \left(\frac{2.4}{b}+1\right)\frac{b^3}{2.4} \ \Longleftrightarrow\ 1.6^2+b^2 = b^2+\frac{b^3}{2.4} \ \Longleftrightarrow\ b = \sqrt[3]{1.6^2\cdot 2.4} \approx 1.83154$$

Substituting this value into $f(b) = \left(\frac{2.4}{b}+1\right)\sqrt{1.6^2+b^2}$, the length of the ladder is approximately 5.61879 m.

Remark. f has a local minimum at $b = \sqrt[3]{1.6^2\cdot 2.4}$, since $f'(b) = \frac{-6.144+b^3}{b^2\sqrt{2.56+b^2}}$ changes sign from negative to positive at this point.

4. (4 points) Estimate the value of $\sqrt[4]{82}$ by the Taylor polynomial of order 3 of the function $f(x) = \sqrt[4]{x}$ at center 81. Give an upper bound for the error of the approximation.

Solution. The derivatives and substitution values are

$f(x) = \sqrt[4]{x}$ $\qquad f(81) = 3$

$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4}\cdot\frac{1}{\left(\sqrt[4]{x}\right)^3}$ $\qquad f'(81) = \frac{1}{4}\cdot\frac{1}{3^3}$

$f''(x) = -\frac{3}{4^2}x^{-\frac{7}{4}} = -\frac{3}{4^2}\cdot\frac{1}{\left(\sqrt[4]{x}\right)^7}$ $\qquad f''(81) = -\frac{3}{4^2}\cdot\frac{1}{3^7}$

$f'''(x) = \frac{21}{4^3}x^{-\frac{11}{4}} = \frac{21}{4^3}\cdot\frac{1}{\left(\sqrt[4]{x}\right)^{11}}$ $\qquad f'''(81) = \frac{21}{4^3}\cdot\frac{1}{3^{11}}$

$f^{(4)}(x) = \frac{231}{4^4}x^{-\frac{15}{4}} = \frac{231}{4^4}\cdot\frac{1}{\left(\sqrt[4]{x}\right)^{15}}$

The Taylor polynomial of order 3 with center $x_0 = 81$ is

$$T_3(x) = f(81) + f'(81)(x-81) + \frac{f''(81)}{2!}(x-81)^2 + \frac{f'''(81)}{3!}(x-81)^3 =$$

$$= 3 + \frac{1}{4}\cdot\frac{1}{3^3}(x-81) - \frac{3}{4^2}\cdot\frac{1}{3^7}\cdot\frac{1}{2!}(x-81)^2 + \frac{21}{4^3}\cdot\frac{1}{3^{11}}\cdot\frac{1}{3!}(x-81)^3$$

If $x = 82$ then $f(82) \approx T_3(82)$, that is,

$$\sqrt[4]{82} = f(82) \approx T_3(82) = 3 + \frac{1}{4}\cdot\frac{1}{3^3} - \frac{3}{4^2}\cdot\frac{1}{3^7}\cdot\frac{1}{2!} + \frac{21}{4^3}\cdot\frac{1}{3^{11}}\cdot\frac{1}{3!}$$

Lagrange remainder term: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$, where $n=3$, $x_0=81$, $x=82$, $81<\xi<82$

Taylor's theorem: $f(x) = T_n(x) + R_n(x)$

The error for the approximation $f(x) \approx T_3(x)$ can be estimate from above:

$$|E| = |f(x) - T_3(x)| = |R_3(x)| = \left|\frac{f^{(4)}(\xi)}{4!}(82-81)^4\right| = \left|\frac{231}{4^4}\cdot\frac{1}{\left(\sqrt[4]{\xi}\right)^{15}}\cdot\frac{1}{4!}\cdot 1^4\right|$$

$$= \frac{231}{4^4}\cdot\frac{1}{\left(\sqrt[4]{\xi}\right)^{15}}\cdot\frac{1}{4!} < \frac{231}{4^4}\cdot\frac{1}{3^{15}}\cdot\frac{1}{4!} \approx 2.62025\times 10^{-9}$$

For the upper estimation we use that $81<\xi<82$.

Comparison of the numerical values: $\sqrt[4]{82} \approx 3.009216698$
$T_3(82) \approx 3.009216701$

Remark. $T(82)$ is a Leibniz series (starting from the second term), so an upper bound for the error for the approximation is $\frac{231}{4^4}\cdot\frac{1}{3^{15}}\cdot\frac{1}{4!}$.

5. (4 points) Estimate the value of $\cos 0.5$ by an appropriate Taylor polynomial with an error less than 10^{-3}.

Solution. Let $f(x) = \cos x$, $x_0 = 0$, $x = 0.5$. We need to determine n such that the error of the approximation $f(x) \approx T_n(x)$ is less than 10^{-3}.

Since $\left|f^{(n)}(x)\right| \leq 1$, then for the error we have

$$|E| = |\cos 0.5 - T_n(0.5)| = |R_n(0.5)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(0.5-0)^{n+1}\right| \leq \frac{0.5^{n+1}}{(n+1)!} < 10^{-3}, \text{ if } n \geq 4.$$

(Here n can be determined by substituting a few values.)

It means that $\cos 0.5 \approx T_4(0.5) = 1 - \frac{0.5^2}{2!} + \frac{0.5^4}{4!} = 1 - \frac{1}{8} + \frac{1}{384} = 0.877604$ is a good approximation.

As a comparison, $\cos(0.5) \approx 0.877583$.

6. (4 points) Find the Taylor series of $f(x) = \frac{1}{(x-2)^2}$ with center -1 and find the radius of convergence.

Solution. If $|x+1| < 3$, then

$$f(x) = \frac{1}{(x-2)^2} = \frac{d}{dx}\left(\frac{1}{2-x}\right) = \frac{d}{dx}\left(\frac{1}{3-(x+1)}\right) = \frac{d}{dx}\left(\frac{1}{3}\frac{1}{1-\frac{x+1}{3}}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{(x+1)^n}{3^{n+1}}\right) =$$

$$= \sum_{n=0}^{\infty}\left(\frac{d}{dx}\frac{(x+1)^n}{3^{n+1}}\right) = \sum_{n=0}^{\infty}\frac{n(x+1)^{n-1}}{3^n} = \sum_{n=0}^{\infty}\frac{(n+1)(x+1)^n}{3^{n+1}}$$

Remark. The Taylor series can also be determined by calculating the derivatives and substituting into the formula $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$.

7. (4 points) Find the Taylor series of $f(x) = \frac{x^2}{\sqrt[5]{32 - x^3}}$ with center 0 and find the radius of convergence.

Solution. If $|x| < \sqrt[3]{32}$, then

$$f(x) = \frac{x^2}{2}\left(1 + \left(-\frac{x}{\sqrt[3]{32}}\right)^3\right)^{-\frac{1}{5}} = \frac{x^2}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{5}}{n} \left(-\frac{x}{\sqrt[3]{32}}\right)^{3n} = \sum_{n=0}^{\infty} \frac{1}{2} \binom{-\frac{1}{5}}{n} \cdot \frac{(-1)^n x^{3n+2}}{32^n}$$