Calculus 1 - 02

Proofs. Inequalities.

Proofs

https://www.whitman.edu/mathematics/higher_math_online/chapter02.html

Direct proof

Since $((P \Longrightarrow R) \wedge (R \Longrightarrow Q)) \implies (P \Longrightarrow Q)$ is always true (it is a tautology), we can prove $P \Longrightarrow Q$ by proving $P \Longrightarrow R$ and $R \Longrightarrow Q$ where R is any other proposition.

Example

Inequality of arithmetic and geometric means:

If $a, b \geq 0$ then $\sqrt{a\,b} \leq \frac{a+b}{2}$ and equality holds if and only if $a = b$.

Proof: $\frac{a+b}{2} \geq \sqrt{a\,b} \iff (a+b)^2 \geq 4\,a\,b \iff a^2 - 2\,a\,b + b^2 \geq 0 \iff (a-b)^2 \geq 0$, which always holds.

Indirect proof

There are two methods of indirect proof: proof of the contrapositive and proof by contradiction. They both start by assuming the denial of the conclusion.

Proof of the contrapositive

We can prove $P \implies Q$ by proving its **contrapositive**, $\neg Q \implies \neg P$. We have seen that these are logically equivalent. In the proof we assume that Q is false and try to prove that P is false.

Example. If $a\,b$ is even then either a or b is even.
Proof. Assume both a and b are odd. Since the product of odd numbers is odd, then $a\,b$ is odd.

Proof by contradiction

To prove a statement P by contradiction we assume $\neg P$ and derive a statement that is known to be false. This means P must be true.

If we want to prove a statement of the form $P \implies Q$ then we assume that P is true and Q is false (since $\neg(P \Longrightarrow Q) \equiv \neg(\neg P \vee Q) \equiv P \wedge \neg Q$) and try to derive a statement known to be false.
This statement need not be $\neg P$, this is the difference between proof by contradiction and proof of the contrapositive.

Examples

In the following two examples we will use the **fundamental theorem of arithmetic** also known as **unique factorization theorem** which states that every integer greater than 1 can be factored uniquely as a product of primes, up to the order of factors.

1) Theorem: There are infinitely many primes.

Proof.

1) Assume there are only finitely many primes $p_1, p_2, \ldots, p_k$ and let $n = p_1 p_2 \ldots p_k + 1$.
2) Then n is not divisible by any of the primes $p_1, p_2, \ldots, p_k$ since the remainder is always 1.
   It means that
   - n is either another prime or
   - it has a prime factor different from $p_1, p_2, \ldots, p_k$.
3) This is a contradiction since we started from the fact that there are exactly k primes
   and then came to the conclusion that there must be at least one more prime.
   It means that there are infinitely many primes.

2) $\sqrt{3}$ is irrational.

Proof.

1) Assume indirectly that $\sqrt{3}$ is rational. Then is can be written in the form $\sqrt{3} = \frac{a}{b}$ where a, b are integers and $b \neq 0$. From this we get that $3\,b^2 = a^2$.
2) Consider the exponent of 3 in the prime factorization of both sides.
   Since in the prime factorization of a square number all exponents are even, it means that
   - the exponent of 3 is odd on the left-hand side and
   - even on the right-hand side.
3) However, this contradicts the unique factorization theorem, so $\sqrt{3}$ is irrational.

Induction

Let $P(n)$ denote a statement that depends on the natural number n.
A proof by induction consists of two cases.
1) The **base case** (or basis) proves that $P(n_0)$ is true without assuming any knowledge of other cases.
2) The **induction step** proves that if $P(k)$ is true for any natural number k, then $P(k+1)$ must also be true.

From these two steps it follows that $P(n)$ holds for all natural numbers $n \geq n_0$.

Examples

1. Prove by induction that for every positive integer n the following statement holds:

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

Solution:

1. Base case: the statement is true for $n = 1$ since $1 = \frac{1 \cdot 2}{2}$.

2. Induction step:

a) Assume that the statement holds for $n = k$, that is,

$$\mathbf{1 + 2 + \ldots + k = \frac{k(k+1)}{2}}$$ (this is the induction hypothesis).

b) Using this, we prove that the statement holds for $n = k + 1$, that is,

$$1 + 2 + \ldots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}.$$

Using the induction hypothesis 2. a) we get:

$$\mathbf{1 + 2 + \ldots + k} + (k + 1) = \mathbf{\frac{k(k+1)}{2}} + (k + 1) = \frac{k(k + 1) + 2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}$$

So the statement is also true for $n = k + 1$. Thus, by the principle of induction, the statement holds for all positive integers n (that is, $n_0 = 1$)

2. Prove by induction that $3^n > 2^n + 7n$ for all positive integers $n \geq n_0$.
Find the smallest such positive integer n_0.

Solution: Let $P(n)$ denote the above statement. Then

$P(1)$ is false, since	$3^1 < 2^1 + 7 \cdot 1$	$P(4)$ is true:	$3^4 > 2^4 + 7 \cdot 4$
$P(2)$ is false, since	$3^2 < 2^2 + 7 \cdot 2$	$P(5)$ is true:	$3^5 > 2^5 + 7 \cdot 5$
$P(3)$ is false, since	$3^3 < 2^3 + 7 \cdot 3$	$P(6)$ is true:	$3^6 > 2^6 + 7 \cdot 6$ etc.

We prove by induction that the statement holds for all integers $n \geq 4 = n_0$.

1. Base case: The statement holds for $n = 4$ since $3^4 > 2^4 + 7 \cdot 4$.
2. Induction step:

a) Assume that the statement holds for $n = k$, that is, $\mathbf{3^k > 2^k + 7k}$.

b) Using this, we prove that the statement holds for $n = k + 1$, that is, $3^{k+1} > 2^{k+1} + 7(k + 1)$.

Using the induction hypothesis 2. a) we get:

$$3^{k+1} = 3 \cdot \mathbf{3^k > } 3 \cdot \mathbf{(2^k + 7k)} =$$
$$= 3 \cdot 2^k + 3 \cdot 7k =$$
$$= (2 + 1) \cdot 2^k + (2 + 1) \cdot 7k =$$
$$= 2 \cdot 2^k + 2^k + 2 \cdot 7k + 7k =$$
$$= 2^{k+1} + 7k + \mathbf{2^k + 2 \cdot 7k >} 2^{k+1} + 7k + \mathbf{0 + 7} =$$
$$= 2^{k+1} + 7(k + 1)$$

So the statement is also true for $n = k + 1$. Thus, by the principle of induction, the statement holds for all integers $n \geq 4$.

Inequalities

Triangle inequality

$|a+b| \leq |a|+|b|$

Proof. Since both sides are nonnegative, then taking the squares of both sides is an equivalent transformation:

$|a+b| \leq |a|+|b| \iff a^2+2ab+b^2 \leq a^2+2|ab|+b^2 \iff 2ab \leq 2|ab|$

This is always true since $x \leq |x|$ for all $x \in \mathbb{R}$.

Inequality of the arithmetic and geometric means

If $a_1, a_2, \ldots a_n \geq 0$ then $\sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1+a_2+\ldots+a_n}{n}$ and equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Proof: by induction.

a) The statement holds for $n = 2$ (see direct proof above): $\frac{a_1+a_2}{2} \geq \sqrt{a_1 a_2}$.

b) We prove that if the statement is true for n then it is also true for $2n$.
For this, divide the arbitrarily fixed $2n$ numbers into two groups of n.
Apply the induction hypothesis for these two groups and then apply part a) for case $n = 2$.

$$\frac{a_1+\ldots+a_{2n}}{2n} = \frac{1}{2}\left(\frac{a_1+\ldots+a_n}{n} + \frac{a_{n+1}+\ldots+a_{2n}}{n}\right) \geq \frac{1}{2}\left(\sqrt[n]{a_1 \ldots a_n} + \sqrt[n]{a_{n+1} \ldots a_{2n}}\right) \geq \sqrt[2n]{a_1 \ldots a_{2n}}.$$

Thus, the statement holds for $n = 2^k$.

c) Using a kind of reverse induction, we prove that if the statement holds for $(n+1)$ then it is also true for n and thus it holds for all positive integers.
Let $a_{n+1} = \frac{a_1+\ldots+a_n}{n} = A_n$ and apply the statement for the $(n+1)$ numbers $a_1, \ldots, a_n, a_{n+1}$.
With equivalent steps, we get

$$A_n = \frac{a_1+\ldots+a_n+A_n}{n+1} \geq \sqrt[n+1]{a_1 \ldots a_n A_n} \iff A_n^{n+1} \geq a_1 \ldots a_n A_n \iff A_n^n \geq a_1 \ldots a_n \iff A_n \geq \sqrt[n]{a_1 \ldots a_n}.$$

d) Finally, we prove the equality part of the theorem.
If $a_1 = \ldots a_n = a$ then the equality obviously holds since $\frac{a_1+\ldots+a_n}{n} = a = \sqrt[n]{a_1 \ldots a_n}$.
Now suppose that for example $a_1 \neq a_2$. Using that in this case $\frac{a_1+a_2}{2} > \sqrt{a_1 a_2}$, we get

$$\frac{a_1+a_2+a_3+\ldots+a_n}{n} = \frac{\frac{a_1+a_2}{2}+\frac{a_1+a_2}{2}+a_3+\ldots+a_n}{n} \geq$$

$$\geq \sqrt[n]{\left(\frac{a_1 + a_2}{2}\right)^2 a_3 \ldots a_n} > \sqrt[n]{\left(\sqrt{a_1 a_2}\right)^2 a_3 \ldots a_n} = \sqrt[n]{a_1 \ldots a_n}.$$

HM-GM-AM-QM inequalities

The inequalities between the **harmonic mean**, **geometric mean**, **arithmetic mean** and **quadratic mean** of the positive real numbers $a_1, a_2, \ldots, a_n$:

$$0 < \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}}$$

Equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Bernoulli's inequality

$(1 + x)^n \geq 1 + n x$ where $x \geq -1$ and n is a positive integer.

Proof: By induction.

1) For $n = 1$: $1 + x \leq 1 + x$.

2) Assume that $(1 + x)^n \geq 1 + n x$ and multiply both sides by $1 + x \geq 0$:

$$(1 + x)^{n+1} \geq (1 + n x) \cdot (1 + x) = 1 + (n + 1) x + n x^2 \geq 1 + (n + 1) x.$$

Exercises

Induction

Prove by induction that the following statements hold for $n \geq n_0$. Find the smallest such positive integer n_0.

1) $1 + 3 + 5 + \ldots + (2 n - 1) = n^2$

2) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n + 1)(2 n + 1)}{6}$

3) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n + 1)}{2}\right)^2$

4) $\sum_{k=1}^{n} k(k + 1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n \cdot (n + 1) = \frac{n(n + 1)(n + 2)}{3}$

5) $\sum_{k=1}^{n} k \cdot k! = (n + 1)! - 1$

6) $\frac{(2 n)!}{(n!)^2} < 4^{n-1}$

Inequalities

1. Prove that

   a) $x^2 + \frac{1}{x^2} \geq 2$ if $x \neq 0$ b) $\frac{x^2}{1 + x^4} \leq \frac{1}{2}$

2. Prove that if $a, b, c > 0$ then

   a) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$ b) $\frac{a^2}{b\,c} + \frac{b^2}{a\,c} + \frac{c^2}{a\,b} \geq 3$

3. Prove that $n! < \left(\frac{n+1}{2}\right)^n$ if $n \geq 2$.

4. What is the maximum of $x\,y$ if $x, y \geq 0$ and

   a) $x + y = 10$; b) $2\,x + 3\,y = 10$?

5. Calculate the maximum value of the function $x^2 \cdot (1 - x)$ for $x \in [0, 1]$.

6. Which rectangular box has the greatest volume among the ones with given surface area?

7. What is the maximum value of $a^3\,b^2\,c$, if a, b, c are non-negative and $a + 2\,b + 3\,c = 5$?

8. Using Bernoulli's inequality, prove that there exists a positive integer n such that

   a) $0.9^n < \frac{1}{100}$ b) $\sqrt[n]{2} < 1.01$ c) $\sqrt[n]{0.1} > 0.9$

Solutions

1. a) Apply the AM-GM inequality for $a_1 = x^2$ and $a_2 = \frac{1}{x^2}$. b) It follows from case a).

2. a) Apply the AM-GM inequality for $x_1 = \frac{a}{b}$, $x_2 = \frac{b}{c}$, $x_3 = \frac{c}{a}$.

   b) Apply the AM-GM inequality for $x_1 = \frac{a^2}{b\,c}$, $x_2 = \frac{b^2}{a\,c}$, $x_3 = \frac{c^2}{a\,b}$.

3. Apply the AM-GM inequality for $a_1 = 1$, $a_2 = 2$, ..., $a_n = n$.

4. a) Apply the AM-GM inequality for $x \geq 0$ and $y \geq 0$:

   $$\sqrt{x\,y} \leq \frac{x + y}{2} = \frac{10}{2} = 5 \Longrightarrow x\,y \leq 25$$

   and equality holds if and only if $x = y$. Since $x + y = 10$ then $2\,x = 10 \Longrightarrow x = 5$,
   so the maximum of $x\,y$ is 25 if $x = y = 5$.

   b) Apply the AM-GM inequality for $2\,x \geq 0$ and $3\,y \geq 0$: $\sqrt{2\,x \cdot 3\,y} \leq \frac{2\,x + 3\,y}{2} = \frac{10}{2} = 5 \Longrightarrow x\,y \leq \frac{25}{6}$

and equality holds if and only if $2x = 3y$. Since $2x + 3y = 10$ then $4x = 10 \implies x = \frac{5}{2}$,
so the maximum of xy is $\frac{25}{6}$ if $x = \frac{5}{2}$, $y = \frac{5}{3}$.

5. Apply the AM-GM inequality for $a_1 = a_2 = x \geq 0$, $a_3 = 2 - 2x \geq 0$:

$$\sqrt[3]{x \cdot x \cdot (2 - 2x)} \leq \frac{x + x + (2 - 2x)}{3} = \frac{2}{3} \implies x^2(1 - x) \leq \frac{4}{27}$$

and equality holds if and only if $x = 2 - 2x$, that is, $x = \frac{2}{3}$.

The maximum of the function $f(x) = x^2(1 - x)$ on $[0, 1]$ is $f\left(\frac{2}{3}\right) = \frac{4}{27}$.

6. The surface area and volume of a box with dimensions x, y, z are
$A = 2(xy + xz + yz)$, $V = xyz$. Let us apply the AM-GM inequality for $xy > 0$, $xz > 0$, $yz > 0$:
$\frac{A}{6} = \frac{xy + xz + yz}{3} \geq \sqrt[3]{xy \cdot xz \cdot yz} = \sqrt[3]{(xyz)^2} = V^{\frac{2}{3}}$ and equality holds if and only if $xy = xz = yz$
from where $x = y = z$, that is, the box is a cube.

7. Apply the AM-GM inequality for the nonnegative numbers $\frac{a}{3}, \frac{a}{3}, \frac{a}{3}, b, b, 3c$:

$$\sqrt[6]{\frac{a}{3} \cdot \frac{a}{3} \cdot \frac{a}{3} \cdot b \cdot b \cdot 3c} \leq \frac{\frac{a}{3} + \frac{a}{3} + \frac{a}{3} + b + b + 3c}{6} = \frac{a + 2b + 3c}{6} = \frac{5}{6} \implies a^3 b^2 c \leq 9 \cdot \left(\frac{5}{6}\right)^6$$

and equality holds if and only if $\frac{a}{3} = b = 3c$. Then substituting $a = 9c$, $b = 3c$ into $a + 2b + 3c = 5$
we get $a = \frac{5}{2}$, $b = \frac{5}{6}$, $c = \frac{5}{18}$, so for these values the maximum of $a^3 b^2 c$ is $9 \cdot \left(\frac{5}{6}\right)^6$.

8. a) $0.9^n < \frac{1}{100} \iff 100 < \left(\frac{10}{9}\right)^n = \left(1 + \frac{1}{9}\right)^n$. Applying Bernoulli's inequality $(1 + x)^n \geq 1 + nx$ with $x = \frac{1}{9}$,
we get $\left(1 + \frac{1}{9}\right)^n \geq 1 + \frac{n}{9}$. If $1 + \frac{n}{9} > 100$ then $n > 891$, so in this case $\left(1 + \frac{1}{9}\right)^n > 100$ also holds.
Remark: Solving the inequality for $n \in \mathbb{N}$, we get that $n \geq 44$.

b) $\sqrt[n]{2} < 1.01 \iff 1.01^n > 2$. Applying Bernoulli's inequality $(1 + x)^n \geq 1 + nx$ with $x = 0.01$,
we get $(1 + 0.01)^n \geq 1 + 0.01n$. If $1 + 0.01n > 2$ then $n > 100$, so in this case $1.01^n > 2$ also holds.
Remark: Solving the inequality for $n \in \mathbb{N}$, we get that $n \geq 70$.

c) $\sqrt[n]{0.1} > 0.9 \iff \frac{1}{10} > \left(\frac{9}{10}\right)^n \iff \left(\frac{10}{9}\right)^n = \left(1 + \frac{1}{9}\right)^n > 10$. Applying Bernoulli's inequality with $x = \frac{1}{9}$,
we get $\left(1 + \frac{1}{9}\right)^n \geq 1 + \frac{n}{9}$. If $1 + \frac{n}{9} > 10$ then $n > 81$, so in this case $\left(1 + \frac{1}{9}\right)^n > 10$ also holds.
Remark: Solving the inequality for $n \in \mathbb{N}$, we get that $n \geq 22$.