Calculus 1 - 07

Numerical series

Definition

Definition. Suppose that (a_n) is a sequence and define the sequence of **partial sums** as

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots a_n.$$

If (s_n) is convergent, then the **numerical series** $\sum_{n=1}^{\infty} a_n$ is convergent,

and its sum is $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = s \in \mathbb{R}$.

Examples

1. a) $\sum_{k=1}^{\infty} 1 = ?$ **b)** $\sum_{k=1}^{\infty} (-1)^{k+1} = ?$

Solution. a) $\sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \ldots = \infty$

Here $s_n = \sum_{k=1}^{n} 1 = n \implies \lim_{n\to\infty} s_n = \infty \implies$ the series is divergent (and its sum is infinity).

b) $\sum_{k=1}^{\infty} (-1)^{k+1} = 1 - 1 + 1 - 1 + \ldots + (-1)^k + \ldots$

Here $s_{2k+1} = 1 \longrightarrow 1$ and $s_{2k} = 0 \longrightarrow 0$, so (s_n) has two limit points.

$\implies$ The series is divergent (and its sum doesn't exist).

2. $$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^k = \lim_{n\to\infty}\left(\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^n\right) = \lim_{n\to\infty}\frac{1}{2}\cdot\frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1} = \frac{1}{2}\cdot\frac{0-1}{-\frac{1}{2}} = 1,$$

so the series is convergent.

A telescoping series

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)} = \lim_{n\to\infty}\left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}\right) =$$

$$= \lim_{n\to\infty}\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \ldots + \frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1,$$ so the series is convergent.

The harmonic series

Theorem. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof. $s_{2^n} = \sum_{k=1}^{2^n} \frac{1}{k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right) \geq$

$$\geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \ldots + 2^{n-1} \cdot \frac{1}{2^n} = 1 + \frac{n}{2} \xrightarrow{n\to\infty} \infty, \text{ so } \lim_{n\to\infty} s_{2^n} = \infty.$$

If $n > 2^k$ then $s_n \geq s_{2^k}$, so $\lim_{n\to\infty} s_n = \infty$ and therefore $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Remark. The name of the harmonic series comes from the fact that for all $n \geq 2$, a_n is the harmonic mean of a_{n-1} and a_{n+1}, that is,

$$a_n = \frac{2}{\frac{1}{a_{n-1}} + \frac{1}{a_{n+1}}} = \frac{2}{\frac{1}{\frac{1}{n-1}} + \frac{1}{\frac{1}{n+1}}} = \frac{2}{(n-1)+(n+1)} = \frac{1}{n}.$$

The divergence of the series is very slow, for example

$$\sum_{n=1}^{100} \frac{1}{n} \approx 5.18738, \quad \sum_{n=1}^{10^4} \frac{1}{n} \approx 9.78761, \quad \sum_{n=1}^{10^5} \frac{1}{n} \approx 12.0901, \quad \sum_{n=1}^{10^6} \frac{1}{n} \approx 14.3927$$

Remark. If a finite number of terms in a series are omitted or changed then the fact of convergence or divergence doesn't change. However, the sum of a convergent series changes.

The geometric series

Theorem. $1 + q + q^2 + \ldots = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ if $|q| < 1$ and the series is divergent otherwise.

Proof. If $a_n = q^n$ then $s_n = \sum_{k=1}^{n} a_k = \sum_{k=0}^{n} q^k = \begin{cases} \frac{q^{n+1}-1}{q-1} & \text{if } q \neq 1 \\ n+1 & \text{if } q = 1 \end{cases}$

1) If $q = 1$ then $\lim_{n\to\infty} s_n = \infty$.

2) If $q > 1$ then $\lim_{n\to\infty} s_n = \infty$, since $\lim_{n\to\infty} q^{n+1} = \infty$.

3) If $-1 < q < 1$ then $\lim_{n\to\infty} s_n = \frac{1}{1-q}$, since $\lim_{n\to\infty} q^{n+1} = 0$.

4) If $q \leq -1$ then $\lim_{n\to\infty} s_n$ does not exist, since $\lim_{n\to\infty} q^n$ does not exist.

Similarly, $\sum_{n=0}^{\infty} a \cdot q^n = \frac{a}{1-q}$, $\sum_{n=k}^{\infty} a \cdot q^n = \frac{a \cdot q^k}{1-q}$ if $|q| < 1$. $\left(\text{sum} = \frac{\text{first term}}{1 - \text{ratio}}\right)$

Sum and constant multiple

Theorem: Assume $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, $\sum_{n=1}^{\infty} d_n$ is divergent, and $c \in \mathbb{R}\backslash\{0\}$. Then

(1) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(2) $\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$

(3) $\sum_{n=1}^{\infty} (a_n + d_n)$ is divergent

(4) $\sum_{n=1}^{\infty} c\, d_n$ is divergent

Proof. All statements follow from the properties of the sequences.

Example. $\sum_{k=2}^{\infty} \frac{3^{k+1} + 5\,(-2)^{k+3}}{4^k} = ?$

Solution. $\sum_{k=2}^{\infty} \frac{3^{k+1} + 5\,(-2)^{k+3}}{4^k} = \sum_{k=2}^{n} \frac{3 \cdot 3^k - 5 \cdot 8 \cdot (-2)^k}{4^k} = 3 \cdot \sum_{k=2}^{\infty} \left(\frac{3}{4}\right)^k - 40 \cdot \sum_{k=2}^{\infty} \left(-\frac{2}{4}\right)^k =$

$$= 3 \cdot \frac{\left(\frac{3}{4}\right)^2}{1 - \frac{3}{4}} - 40 \cdot \frac{\left(-\frac{1}{2}\right)^2}{1 - \left(-\frac{1}{2}\right)} = \frac{1}{12}$$

The series is the sum of two convergent geometric series.

Cauchy criterion

Theorem: The numerical series $\sum_{n=1}^{\infty} a_n$ converges if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$

such that if $m > n > N$ then $|\, s_m - s_n \,| = \sum_{k=n+1}^{m} a_k = |\, a_{n+1} + a_{n+2} + \ldots + a_m \,| < \varepsilon$.

Proof: It is trivially true, since the Cauchy criterion for number sequences can be applied for (s_n).

Example. Is the series $\sum_{k=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ convergent or divergent? (alternating harmonic series)

Solution. The series is convergent. Let $m > n$ and $m = n + k$. Then

$$|\, s_m - s_n \,| = |\, s_{n+k} - s_n \,| = |\, a_{n+1} + a_{n+2} + \ldots + a_{n+k} \,| = \left| \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n+3}}{n+2} + \frac{(-1)^{n+4}}{n+3} + \ldots + \frac{(-1)^{n+k+1}}{n+k} \right| =$$

$$= \left| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \ldots + \frac{(-1)^{k+1}}{n+k} \right|.$$

Using that $\frac{1}{n+1} - \frac{1}{n+2} > 0$, $\frac{1}{n+2} - \frac{1}{n+3} > 0$ etc. we get the following.

1) If k is even then

$$\left| s_{n+k} - s_n \right| = \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+3} - \frac{1}{n+4}\right) + \ldots + \left(\frac{1}{n+k-1} - \frac{1}{n+k}\right) =$$
$$= \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \ldots - \left(\frac{1}{n+k}\right) < \frac{1}{n+1}$$

2) If k is odd then

$$\left| s_{n+k} - s_n \right| = \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+3} - \frac{1}{n+4}\right) + \ldots + \left(\frac{1}{n+k-2} - \frac{1}{n+k-1}\right) + \frac{1}{n+k} =$$
$$= \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \ldots - \left(\frac{1}{n+k-1} - \frac{1}{n+k}\right) < \frac{1}{n+1}.$$

Then $\left| s_{n+k} - s_n \right| < \frac{1}{n+1} < \varepsilon$ if $n > \frac{1}{\varepsilon} - 1$, so with the choice $N(\varepsilon) \geq \left[\frac{1}{\varepsilon} - 1\right]$ the statement holds.

Later we will see that this is a Leibniz series, so it is convergent.

The *n*th term test

Theorem: If $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

1st proof: Apply the Cauchy criterion with the choice $m = n + 1$. Then

$$\left| s_{n+1} - s_n \right| = \left| a_{n+1} \right| < \varepsilon \text{ if } n > N(\varepsilon), \text{ so } \lim_{n\to\infty} a_n = 0.$$

2nd proof: Let $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, then $s_n = s_{n-1} + a_n \implies a_n = s_n - s_{n-1} \longrightarrow s - s = 0$.

Remark. The theorem can also be stated in the following form:

If $\lim_{n\to\infty} a_n \neq 0$ or if the limit doesn't exist then $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. The condition $\lim_{n\to\infty} a_n = 0$ is necessary but not sufficient for the convergence of $\sum_{n=1}^{\infty} a_n$.

For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent but $\lim_{n\to\infty} \frac{1}{n} = 0$.

Series with nonnegative terms

Theorem. A series with nonnegative terms converges if and only if its partial sums form a bounded sequence.

Proof. If $a_n \geq 0$ for all $n \in \mathbb{N}$ then $s_{n+1} = a_{n+1} + s_n \geq s_n$ for all $n \in \mathbb{N}$, so (s_n) is monotonically increasing.

If $\sum_{n=1}^{\infty} a_n$ converges, then (s_n) converges $\Longrightarrow$ (s_n) is bounded.

If (s_n) is bounded, then (s_n) converges since it is monotonically increasing.

Remark. If $a_n \geq 0$ then $\sum_{n=1}^{\infty} a_n$ either converges or its sum is ∞.

Cauchy Condensation Test

Theorem. Suppose $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2 a_2 + 4 a_4 + 8 a_8 + \ldots$ converges.

Proof. Let $s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$ and $t_n = a_1 + 2 a_2 + 4 a_4 + 8 a_8 + \ldots + 2^n a_{2^n} = \sum_{k=1}^{n} 2^k a_{2^k}$

1) (s_n) is monotonically increasing, since the terms of (a_n) are nonnegative and $n \leq 2^n - 1$ for all $n \in \mathbb{N}^+$ so $s_n \leq s_{2^n - 1}$. Then

$$s_n \leq s_{2^n-1} = \mathbf{a_1} + (\mathbf{a_2 + a_3}) + (\mathbf{a_4 + a_5 + a_6 + a_7}) + \ldots + (a_{2^{n-1}} + \ldots + a_{2^n-1}) \leq$$
$$\leq \mathbf{a_1} + (\mathbf{a_2 + a_2}) + (\mathbf{a_4 + a_4 + a_4 + a_4}) + \ldots + (a_{2^{n-1}} + \ldots + a_{2^{n-1}}) =$$
$$= \mathbf{a_1} + \mathbf{2\, a_2} + \mathbf{4\, a_4} + \ldots + 2^{n-1} a_{2^{n-1}} =$$
$$= \frac{1}{2} (a_1 + 2 a_2 + 4 a_4 + 8 a_8 + \ldots + 2^n a_{2^n}) = t_{n-1}$$

Assume that $\sum_{k=1}^{n} 2^k a_{2^k}$ is convergent $\Longrightarrow$ (t_n) is convergent, so it is bounded $\Longrightarrow$ (s_n) is bounded above since $s_n \leq s_{2^n-1} \leq t_{n-1}$ $\Longrightarrow$ (s_n) is convergent since it is monotonically increasing.

2)
$$s_{2^n} = \mathbf{a_1} + \mathbf{a_2} + (\mathbf{a_3 + a_4}) + (\mathbf{a_5 + a_6 + a_7 + a_8}) + \ldots + (a_{2^{n-1}+1} + \ldots + a_{2^n}) \geq$$
$$\geq \mathbf{\frac{1}{2}\, a_1} + \mathbf{a_2} + (\mathbf{a_4 + a_4}) + (\mathbf{a_8 + a_8 + a_8 + a_8}) + \ldots + (a_{2^n} + \ldots + a_{2^n}) =$$
$$= \mathbf{\frac{1}{2}\, a_1} + \mathbf{a_2} + \mathbf{2\, a_4} + \mathbf{4\, a_8} + \ldots + 2^{n-1} a_{2^n} = \frac{1}{2} t_n$$

Assume that $\sum_{n=1}^{\infty} a_n$ is convergent $\Longrightarrow$ (s_n) is convergent, so it is bounded $\Longrightarrow$ (t_n) is bounded above since $\frac{1}{2} t_n \leq s_{2^n}$ $\Longrightarrow$ (t_n) is convergent since it is monotonically increasing $\Longrightarrow$ $\sum_{k=0}^{\infty} 2^k a_{2^k}$ is convergent.

The p-series (or hyperharmonic series)

Theorem. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof. 1) If $p \leq 0$ then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{|p|} \neq 0$, so by the nth term test, the series diverges.

2) If $p > 0$ then $a_n = \frac{1}{n^p}$ is monotonically decreasing, so the Cauchy condensation theorem is applicable, that is, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and $\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{(2^k)^p}$ are both convergent or both divergent. Then

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{2^{-k}} \cdot \frac{1}{2^{kp}} = \sum_{k=1}^{\infty} \frac{1}{2^{(p-1)k}} = \sum_{k=1}^{\infty} \left(\left(\frac{1}{2}\right)^{p-1}\right)^k.$$

This is a geometric series with ratio $r = \left(\frac{1}{2}\right)^{p-1}$ and it is convergent if and only if

$$|\, r \,| = \left(\frac{1}{2}\right)^{p-1} < 1 \iff p - 1 > 0 \iff p > 1.$$

Examples

1. Is the series $\sum_{n=n_1}^{\infty} \frac{1}{n \cdot \log_2 n}$ convergent or divergent?

Solution. The sequence $a_n = \frac{1}{n \cdot \log_2 n}$ is monotonic decreasing and the terms are nonnegative, so the Cauchy Condensation Test can be applied.

$\sum_{k=k_1}^{\infty} 2^k \cdot a_{2^k} = \sum_{k=k_1}^{\infty} 2^k \cdot \frac{1}{2^k \cdot \log_2(2^k)} = \sum_{k=k_1}^{\infty} \frac{1}{k}$, this the harmonic series which is divergent

$\Longrightarrow$ the series $\sum_{n=n_1}^{\infty} a_n$ is divergent.

2. Show that $\sum_{n=n_1}^{\infty} \frac{1}{n \cdot (\log_2 n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Solution. If $p > 0$ then the sequence $a_n = \frac{1}{n \cdot (\log_2 n)^p}$ is monotonic decreasing and the terms are nonnegative, so the Cauchy Condensation Test can be applied.

$\sum_{k=k_1}^{\infty} 2^k \cdot a_{2^k} = \sum_{k=k_1}^{\infty} 2^k \cdot \frac{1}{2^k \cdot \log_2(2^k)^p} = \sum_{k=k_1}^{\infty} \frac{1}{k^p}$, this the p-series which converges if $p > 1$ and diverges if $p \leq 1$.

If $p \leq 0$ then for example the comparison test can be used to show divergence (see later).

Then $a_n \geq \frac{1}{n}$ and $\sum_{n=n_1}^{\infty} \frac{1}{n}$ diverges $\Longrightarrow \sum_{n=n_1}^{\infty} a_n$ also diverges.

3. Is the series $\sum_{n=n_1}^{\infty} \frac{1}{n \cdot \log_2 n \cdot \log_2 \log_2 n}$ convergent or divergent?

Solution. The sequence $a_n = \frac{1}{n \cdot \log_2 n \cdot \log_2 \log_2 n}$ is monotonic decreasing and the terms are nonnegative, so the Cauchy Condensation Test can be applied.

$\sum_{k=k_1}^{\infty} 2^k \cdot a_{2^k} = \sum_{k=k_1}^{\infty} 2^k \cdot \frac{1}{2^k \cdot \log_2(2^k) \cdot \log_2(\log_2(2^k))} = \sum_{k=k_1}^{\infty} \frac{1}{k \cdot \log_2 k}$, this is divergent (see example 1.)

$\Longrightarrow$ the series $\sum_{n=n_1}^{\infty} a_n$ is also divergent.