Calculus 1 - 08

Comparison test

Theorem. Assume that $0 \le c_n \le a_n \le b_n$ for $n > N$ where N is some fixed integer. Then

(1) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(2) If $\sum_{n=1}^{\infty} c_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof. Denote by s_n^a, s_n^b, s_n^c the nth partial sums of the numerical series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ respectively.

(1) **1st proof.** We use the Cauchy criterion. Let $\varepsilon > 0$ be fixed, then by the convergence of $\sum_{n=1}^{\infty} b_n$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if $m > n > N(\varepsilon)$, then $\left| s_m^b - s_n^b \right| < \varepsilon$, so if $m > n > \max\{N, N(\varepsilon)\}$ then $\left| s_m^a - s_n^a \right| = \sum_{k=n+1}^{m} a_k \le \sum_{k=n+1}^{m} b_k = \left| s_m^b - s_n^b \right| < \varepsilon$, so $\sum_{n=1}^{\infty} a_n$ is convergent.

2nd proof. Changing finitely many terms does not affect the convergence or divergence of a series, so it may be assumed that $0 \le a_n \le b_n$ holds for all $n \in \mathbb{N}$. (If the series does not start at $n = 1$ then it can be reindexed.)

From the condition $\begin{cases} a_1 \le b_1 \\ a_2 \le b_2 \\ \dots \\ a_n \le b_n \end{cases} \implies s_n^a = a_1 + a_2 + \dots + a_n \le b_1 + b_2 + \dots + b_n = s_n^b.$

Assume that $\sum_{n=1}^{\infty} b_n$ is convergent $\implies (s_n^b)$ is bounded $\implies (s_n^a)$ is bounded

$\implies (s_n^a)$ is convergent since it is monotonically increasing $\implies \sum_{n=1}^{\infty} a_n$ is convergent.

(2) (s_n^c) is monotonically increasing if $n > N$ and not bounded, so $s_n^a - s_N^a > s_n^c - s_N^c \longrightarrow \infty$ and thus $s_n^a \longrightarrow \infty$.

Remark. The convergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ can be investigated easily with the comparison test for $p \le 1$ and $p \ge 2$.

- If $p \le 1$ then $0 < \frac{1}{n} \le \frac{1}{n^p}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent.

- If $p = 2$ then $\frac{1}{n^2} \leq \frac{2}{n(n+1)}$ for all $n \in \mathbb{N}^+$ and $\sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.
- If $p > 2$ then $0 < \frac{1}{n^p} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

Remark. Leonhard Euler proved in 1734 that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Examples

1) Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{2n+1} = \sum_{n=1}^{\infty} a_n$.

Solution. Here infinitely many terms are omitted from the harmonic series. By the comparison test we show that this series is still divergent.

$$a_n = \frac{1}{2n+1} > \frac{1}{2n+n} = \frac{1}{3n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

2) Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{n+2}{3n^4+5} = \sum_{n=1}^{\infty} a_n$.

Solution. $$a_n = \frac{n+2}{3n^4+5} < \frac{n+2n}{3n^4+0} = \frac{1}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{converges } (p = 3 > 1) \implies \sum_{n=1}^{\infty} a_n \text{ converges.}$$

3) Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{2n^2-32}{n^3+8} = \sum_{n=1}^{\infty} a_n$.

Solution. If $n \geq 4$ then the terms of the series are positive. By the comparison test we show that the series diverges. If $n \geq 6$ then $n^2 > 32$, so

$$a_n = \frac{2n^2-32}{n^3+8} > \frac{2n^2-n^2}{n^3+8n^3} = \frac{1}{9n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{9n} = \frac{1}{9} \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges} \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

4) Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n+3^{n+1}}{2^{2n+3}+5} = \sum_{n=1}^{\infty} a_n$.

Solution. $$a_n = \frac{2^n+3\cdot 3^n}{8\cdot 4^n+5} < \frac{3^n+3\cdot 3^n}{8\cdot 4^n+0} = \frac{1}{2}\left(\frac{3}{4}\right)^n \quad \text{and}$$

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{4}\right)^n \text{ is a convergent geometric series } \left(q = \frac{3}{4},\ |q| < 1\right) \implies \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Error estimation for series with nonnegative terms

Remark. Usually we don't know the limit $s=\sum_{n=1}^{\infty}a_n=\lim_{n\to\infty}\sum_{k=1}^{n}a_k=\lim_{n\to\infty}s_n$ but if n is large then s_n gives an estimation of s. The error for the approximation $s\approx s_n$ is $|E|=|s-s_n|$.
If $0\le a_k\le b_k$ for $k\ge n$ then the error can be estimated with the comparison test:

$$|E|=|s-s_n|=s-s_n=\sum_{k=1}^{\infty}a_k-\sum_{k=1}^{n}a_k=\sum_{k=n+1}^{\infty}a_k\le\sum_{k=n+1}^{\infty}b_k.$$

Here $s_n\le s$, since (s_n) is monotonically increasing.

Example. Show that the series $\sum_{n=1}^{\infty}\frac{1}{n!}$ is convergent and estimate the error if the sum of the series is approximated by the sum of the first 6 terms ($s\approx s_6$).

Solution. Estimate the terms from above by the terms of a convergent series:

$$\frac{1}{n!}=\frac{1}{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot2\cdot1}\le\frac{1}{n(n-1)\cdot1\cdot\ldots\cdot1\cdot1}=\frac{1}{n^2-n}\le\frac{1}{n^2-\frac{n^2}{2}}=\frac{2}{n^2}.$$

Since $\sum_{n=0}^{\infty}\frac{2}{n^2}$ converges then by the comparison test $\sum_{n=0}^{\infty}\frac{1}{n!}$ also converges.
Error estimation for the approximation $s\approx s_n$:

$$|E|=|s-s_n|=\sum_{k=n+1}^{\infty}a_k=\frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\frac{1}{(n+2)(n+3)(n+4)}+\ldots\right)\le$$
$$\le\frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)^2}+\frac{1}{(n+2)^3}+\ldots\right)=\frac{1}{(n+1)!}\sum_{k=0}^{\infty}\left(\frac{1}{n+2}\right)^k=$$
$$=\frac{1}{(n+1)!}\cdot\frac{1}{1-\frac{1}{n+2}}=\frac{1}{(n+1)!}\cdot\frac{n+2}{n+1}$$

If $n=6$ then $\left|s-s_n\right|\le\frac{1}{7!}\cdot\frac{8}{7}\approx0.000226757$ and
$s_6=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}\approx2.718\ldots\approx e$ (here 3 digits are accurate).

Absolute convergence

Definition. We say that the numerical series $\sum_{n=1}^{\infty}a_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty}|a_n|$ is convergent.

Example. $\sum_{n=1}^{\infty}a_1q^{n-1}$ is absolutely convergent if $|q|<1$.

Theorem. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is convergent.

Proof. Let $\varepsilon > 0$ be fixed. If $\sum_{n=1}^{\infty} |a_n|$ is convergent then by the Cauchy criterion there exists $N \in \mathbb{N}$ such that if $m > n > N$ then $\left| |a_{n+1}| + |a_{n+2}| \ldots + |a_m| \right| < \varepsilon$. Then for all $m > n > N$

$$|s_m - s_n| = |a_{n+1} + a_{n+2} \ldots + a_m| \leq \left| |a_{n+1}| + |a_{n+2}| \ldots + |a_m| \right| < \varepsilon$$

also holds, so by the Cauchy criterion $\sum_{n=1}^{\infty} a_n$ is convergent.

Consequence. If $|a_n| \leq b_n$ for $n > N$ and $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore also convergent.

Definition. If $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent then it is **conditionally convergent**.

Example. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n} \right) = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)}$ is convergent, since $0 < \frac{1}{2n(2n-1)} \leq \frac{1}{2n \cdot n} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

On the other hand $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent, so the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Rearrangements

Definition. If $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ is a permutation of the natural numbers (that is, every natural number appears exactly once in this sequence) then we say that $\sum_{n=1}^{\infty} a_{\pi(n)}$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$.

Theorem (Riemann rearrangement theorem). Suppose that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent and $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $\sum_{n=1}^{\infty} a_n'$ with partial sums s_n' such that $\liminf s_n' = \alpha$, $\limsup s_n' = \beta$.

Theorem. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then every rearrangement of $\sum_{n=1}^{\infty} a_n$ converges and they all converge to the same sum.

Proof: See W. Rudin: Principles of Mathematical Analysis, page 75:
https://web.math.ucsb.edu/~agboola/teaching/2021/winter/122A/rudin.pdf

Alternating series

Definition. $\sum_{n=1}^{\infty} a_n$ is an alternating series if $a_n a_{n+1} < 0$ for all $n \in \mathbb{N}$.

Theorem (Leibniz). Let (a_n) be a monotonically decreasing sequence of positive numbers such that $a_n \xrightarrow{n\to\infty} 0$. Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots$ is convergent.

Remark. A series with this property is called a Leibniz series.
The theorem is called the alternating series test or Leibniz's test or Leibniz criterion.

Proof. Since $a_n \geq a_{n+1} > 0$ for all $n \in \mathbb{N}$ then

$$s_{2n} \leq s_{2n} + (a_{2n+1} - a_{2n+2}) = s_{2n+2} = s_{2n+1} - a_{2n+2} \leq s_{2n+1} = s_{2n-1} - (a_{2n} - a_{2n+1}) \leq s_{2n-1},$$

that is, $0 \leq s_2 \leq s_4 \leq s_6 \leq s_8 \leq \ldots \leq s_7 \leq s_5 \leq s_3 \leq s_1 = a_1$.

So (s_{2n}) is monotonically increasing and bounded above $\Longrightarrow$ it is convergent,
and (s_{2n+1}) is monotonically decreasing and bounded below $\Longrightarrow$ it is convergent.

Since $s_{2n+1} - s_{2n} = a_{2n+1} \xrightarrow{n\to\infty} 0$ then $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_n \Longrightarrow$ the series is convergent.

(Or, by the Cantor axiom $\bigcap_{n=1}^{\infty} [s_{2n}, s_{2n-1}]$ is not empty and since $s_{2n-1} - s_{2n} = a_{2n} \xrightarrow{n\to\infty} 0$ then is has only one element which is the limit of (s_n).)

Error estimation:

Let $s = \lim_{n\to\infty} s_n$. If n is odd then $s_{n+1} \leq s \leq s_n$ and if n is even then $s_n \leq s \leq s_{n+1}$.
In both cases the error for the approximation $s \approx s_n$ is

$$|E| = |s - s_n| \leq |s_{n+1} - s_n| = a_{n+1}.$$

Examples

1. The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent, since $a_n = \frac{1}{n}$ is monotonically decreasing and $a_n \longrightarrow 0$.

2. Is the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt[3]{2n+1}} = \sum_{n=1}^{\infty} (-1)^{n+1} c_n$ convergent?

Solution. Since $c_n = \frac{1}{\sqrt[3]{2n+1}}$ is monotonically decreasing and $c_n \longrightarrow 0$ then this is a Leibniz series so it is convergent.

3. Is the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt[n]{2n+1}} = \sum_{n=1}^{\infty} (-1)^{n+1} c_n$ convergent?

Solution. Since $\frac{1}{\sqrt[n]{3} \cdot \sqrt[n]{n}} = \frac{1}{\sqrt[n]{2n+n}} \le \frac{1}{\sqrt[n]{2n+1}} = c_n \le \frac{1}{\sqrt[n]{0+1}} = 1$

and $\frac{1}{\sqrt[n]{3} \cdot \sqrt[n]{n}} \longrightarrow \frac{1}{1 \cdot 1} = 1$ then by the sandwich theorem $\lim_{n\to\infty} c_n = 1$.

So $\lim_{n\to\infty} (-1)^{n+1} c_n$ doesn't exist, and thus by the nth term test the series diverges.

4. Is the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n^2+2} = \sum_{n=1}^{\infty} (-1)^{n+1} c_n$ convergent?

Solution. 1) $0 < c_n = \frac{n+1}{n^2+2} = \frac{\frac{1}{n} + \frac{1}{n^2}}{1 + \frac{2}{n^2}} \longrightarrow \frac{0+0}{1+0} = 0.$

2) It is not obvious that (c_n) is monotonically decreasing, since both the numerator and the denominator increases.

$$c_{n+1} \le c_n \iff \frac{(n+1)+1}{(n+1)^2+2} \le \frac{n+1}{n^2+2}$$
$$\iff (n+2)\left(n^2+2\right) \le (n+1)\left(n^2+2n+3\right)$$
$$\iff n^3+2n^2+2n+4 \le n^3+n^2+2n^2+2n+3n+3$$
$$\iff 0 \le n^2+3n-1 \quad \text{and this is true for all } n \in \mathbb{N}.$$

Since the steps are equivalent then $c_{n+1} \le c_n$ also holds for all $n \in \mathbb{N}$, so (c_n) is monotonically decreasing. Then by the Leibniz criterion the series converges.

Remark. If the sum of the series is approximated by s_{100} then the error is

$$|E| = |s - s_{100}| \le c_{101} = \frac{101+1}{101^2+2}.$$

Root test (Cauchy)

Theorem (Root test): Assume that $a_n > 0$ and $\limsup \sqrt[n]{a_n} = R$. Then

(1) if $R < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent;

(2) if $R > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof. (1) • Suppose that $R < 1$, then there exists $\varepsilon > 0$ such that $R + \varepsilon < 1$.

• By the definition of the limsup, for this ε there exists $N \in \mathbb{N}$ such that if $n > N$ then $\sqrt[n]{a_n} < R + \varepsilon$, since if $\sqrt[n]{a_n} \ge R + \varepsilon$ would hold for infinitely many n then this subsequence would have a limit point greater than R.

- Thus $a_n \leq (R+\varepsilon)^n$ if $n > N$, and since $\sum_{n=1}^{\infty} (R+\varepsilon)^n$ is a convergent geometric series then by the comparison test, $\sum_{n=1}^{\infty} a_n$ is also convergent.

(2) • Suppose that $R > 1$, then there exists $\varepsilon > 0$ and a subsequence of $\sqrt[n]{a_n}$ such that $\sqrt[n_k]{a_{n_k}} \geq R - \varepsilon > 1$.

- Then for the terms of this subsequence $a_{n_k} \geq (R-\varepsilon)^{n_k} > 1$
$\Longrightarrow \lim_{n_k \to \infty} a_{n_k} \neq 0 \Longrightarrow \lim_{n \to \infty} a_n \neq 0 \Longrightarrow$ the series is divergent by the nth term test.

Consequence. Assume $\limsup \sqrt[n]{|a_n|} = R$. Then

(1) if $R < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent, since it is absolutely convergent;

(2) if $R > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent, since if $\lim_{n\to\infty} |a_n| \neq 0$, then $\lim_{n\to\infty} a_n \neq 0$.

Remark. If $R = 1$ then we don't know anything about the convergence of the series, for example

1) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\sqrt[n]{\frac{1}{n}} \longrightarrow 1$

2) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and $\sqrt[n]{\frac{1}{n^2}} \longrightarrow 1$

Ratio test (D'Alambert)

Theorem (Ratio test): Assume that $a_n > 0$. Then

(1) if $\limsup \frac{a_{n+1}}{a_n} < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent;

(2) if $\liminf \frac{a_{n+1}}{a_n} > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof. (1) • Suppose that $R = \limsup \frac{a_{n+1}}{a_n} < 1$, then similarly as in the previous proof, there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that if $n \geq N$ then $\frac{a_{n+1}}{a_n} < R + \varepsilon < 1$.

- Thus $a_{N+1} < (R+\varepsilon)\, a_N$

$$a_{N+2} < (R+\varepsilon)\, a_{N+1} < (R+\varepsilon)^2\, a_N$$

...

$$a_{n+1} < (R+\varepsilon)\, a_n = (R+\varepsilon)^{n+1-N}\, a_N = \frac{a_N}{(R+\varepsilon)^N} \cdot (R+\varepsilon)^{n+1}$$

so we can apply the comparison test similarly as in the proof of the root test.

(2) • Suppose that $\liminf \frac{a_{n+1}}{a_n} > 1$, then there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that if $n \geq N$ then $\frac{a_{n+1}}{a_n} > R - \varepsilon > 1$.

- Since $a_n > 0$ then $a_{n+1} > a_n$, so (a_n) is monotonic increasing $\Longrightarrow \lim\limits_{n\to\infty} a_n \neq 0$

  $\Longrightarrow$ the series is divergent by the nth term test.

Consequence. Assume $a_n \neq 0$ for all $n \in \mathbb{N}$. Then

(1) if $\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent, since it is absolutely convergent;

(2) if $\liminf \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent, since if $\lim\limits_{n\to\infty} |a_n| \neq 0$, then $\lim\limits_{n\to\infty} a_n \neq 0$.

Remark. If $\limsup \frac{a_{n+1}}{a_n} = 1$ or $\liminf \frac{a_{n+1}}{a_n} = 1$ then we don't know anything about the convergence of the series, for example

1) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent and $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \longrightarrow 1$

2) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} \longrightarrow 1$

Remark. The ratio test is a consequence of the root test and the following theorem.
The proof of this theorem contains a very interesting step.

1) Recall that

- if $x < B$ (or $x \leq B$) **for all** $B > 0$ then $x \leq 0$.

2) Similarly, we can prove $x \leq y$ in the following way:

- if $x \leq B$ **for all** $B > y$ then $x \leq y$.

Theorem. Assume that $a_n > 0$. Then $\liminf \frac{a_{n+1}}{a_n} \leq \liminf \sqrt[n]{a_n} \leq \limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}$.

Proof. 1) We prove that $\limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}$.

Let $\limsup \frac{a_{n+1}}{a_n} = C$ and let $B > C$ be an arbitrary real number.

Then by the definition of the lim sup, there exists $N \in \mathbb{N}$ such that if $k \geq N$ then $\frac{a_{k+1}}{a_k} < B$.

$\Longrightarrow a_{N+1} < B\, a_N, \quad a_{N+2} < B\, a_{N+1} < B^2\, a_N, \quad \ldots$

So if $n > N$ then $a_n < B^{n-N}\, a_N \Longrightarrow \sqrt[n]{a_n} < \sqrt[n]{B^{n-N}}\, \sqrt[n]{a_N} = B \cdot \sqrt[n]{\frac{a_N}{B^N}}$

$\Longrightarrow \limsup \sqrt[n]{a_n} \leq \lim\limits_{n\to\infty} B \cdot \sqrt[n]{\frac{a_N}{B^N}} = B.$

We obtained that the following implication holds **for all** $B > C$:

$\limsup \frac{a_{n+1}}{a_n} < B \Longrightarrow \limsup \sqrt[n]{a_n} \leq B$.

From this it follows that $\limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}$.

2) $\liminf \sqrt[n]{a_n} \leq \limsup \sqrt[n]{a_n}$ is obvious.

3) The proof of $\liminf \frac{a_{n+1}}{a_n} \leq \liminf \sqrt[n]{a_n}$ is similar to case 1).

Consequence. If $a_n > 0$ for all n and $\exists \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \alpha \in \mathbb{R}$ then $\exists \lim\limits_{n\to\infty} \sqrt[n]{a_n} = \alpha$.

Remark. It is a consequence of the previous inequalities that the root test is “stronger” than the ratio test. Consider the series

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots, \text{ where } a_{2k-1} = \frac{1}{2^k} \text{ and } a_{2k} = \frac{1}{3^k},\ k \geq 1.$$

With the root test: • If n is odd, then $\sqrt[n]{a_n} = \sqrt[2k-1]{a_{2k-1}} = \sqrt[2k-1]{\frac{1}{2^k}} \longrightarrow \frac{1}{\sqrt{2}}$ and

• if n is even, then $\sqrt[n]{a_n} = \sqrt[2k]{a_{2k}} = \sqrt[2k]{\frac{1}{3^k}} = \frac{1}{\sqrt{3}}$.

$\Longrightarrow \limsup \sqrt[n]{a_n} = \frac{1}{\sqrt{2}} < 1 \Longrightarrow$ the series is convergent.

With the ratio test: • If n is even, then $\frac{a_{n+1}}{a_n} = \frac{a_{2k+1}}{a_{2k}} = \frac{\frac{1}{2^{k+1}}}{\frac{1}{3^k}} = \frac{3^k}{2^{k+1}} \longrightarrow \infty$ and

• if n is odd, then $\frac{a_{n+1}}{a_n} = \frac{a_{2k}}{a_{2k-1}} = \frac{\frac{1}{3^k}}{\frac{1}{2^k}} = \frac{2^k}{3^k} \longrightarrow 0$.

$\Longrightarrow \limsup \frac{a_{n+1}}{a_n} = \infty > 1$ and $\liminf \frac{a_{n+1}}{a_n} = 0 < 1$

$\Longrightarrow$ the ratio test cannot be used here.

Cauchy product

Definition: The Cauchy product of the series $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ is the series $\sum\limits_{n=0}^{\infty} c_n$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

	a_0	a_1	a_2	a_3	...
b_0	$a_0 b_0$	$a_1 b_0$	$a_2 b_0$	$a_3 b_0$	
b_1	$a_0 b_1$	$a_1 b_1$	$a_2 b_1$		
b_2	$a_0 b_2$	$a_1 b_2$			
b_3	$a_0 b_3$				
...					

Mertens' theorem

Theorem (Mertens). If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent and $\sum_{n=0}^{\infty} b_n$ is convergent, then their Cauchy product is convergent and its sum is $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\, b_{n-k} = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$.

Proof. Let $A = \sum_{n=0}^{\infty} a_n$, $\quad B = \sum_{n=0}^{\infty} b_n$,

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n = \sum_{k=0}^{n} c_k = \sum_{k=0}^{n}\sum_{i=0}^{k} a_i\, b_{k-i}, \quad \beta_n = B_n - B.$$

Then

$$\begin{aligned} C_n &= a_0\, b_0 + (a_0\, b_1 + a_1\, b_0) + (a_0\, b_2 + a_1\, b_1 + a_2\, b_0) + \ldots + (a_0\, b_n + a_1\, b_{n-1} + \ldots + a_n\, b_0) = \\ &= a_0\, B_n + a_1\, B_{n-1} + a_n\, B_{n-2} + \ldots + a_n\, B_0 = \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + a_2(B + \beta_{n-2}) + \ldots + a_n(B + \beta_0) = \\ &= A_n\, B + \boldsymbol{(a_0\, \beta_n + a_1\, \beta_{n-1} + a_2\, \beta_{n-2} + \ldots + a_n\, \beta_0)}. \end{aligned}$$

Let $\boldsymbol{\gamma_n = a_0\, \beta_n + a_1\, \beta_{n-1} + a_2\, \beta_{n-2} + \ldots + a_n\, \beta_0}$.

We have to show that $C_n \longrightarrow A\, B$. Since $A_n\, B \longrightarrow A\, B$, it is enough to show that $\boldsymbol{\lim_{n\to\infty} \gamma_n = 0}$.

Let $\alpha = \sum_{n=0}^{\infty} |\, a_n\,|$. (Here we use that $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.) Let $\varepsilon > 0$ be given.

Since $B = \sum_{n=0}^{\infty} b_n$ then $\beta_n \longrightarrow 0$, so there exists $N \in \mathbb{N}$ such that $|\, \beta_n\,| \le \varepsilon$ if $n \ge N$. In this case

$$\begin{aligned} |\, \gamma_n\,| &\le |\, \beta_0\, a_n + \ldots \beta_N\, a_{n-N}\,| + |\, \beta_{N+1}\, a_{n-N-1} + \ldots + \beta_n\, a_0\,| \le \\ &\le |\, \beta_0\, a_n + \ldots \beta_N\, a_{n-N}\,| + |\, \beta_{N+1}\,| \cdot |\, a_{n-N-1}\,| + \ldots + |\, \beta_n\,| \cdot |\, a_0\,| \le \\ &\le \left|\, \beta_0\, a_n + \ldots \beta_N\, a_{n-N}\,\right| + \varepsilon \cdot \sum_{n=0}^{n-N-1} \left|\, a_n\,\right| \le \\ &\le |\, \beta_0\, a_n + \ldots \beta_N\, a_{n-N}\,| + \varepsilon\, \alpha. \end{aligned}$$

If N is fixed and $n \longrightarrow \infty$ then $|\, \beta_0\, a_n + \ldots \beta_N\, a_{n-N}\,| \longrightarrow 0$ since $a_k \longrightarrow \infty$ as $k \longrightarrow \infty$.
So we get that $\limsup |\, \gamma_n\,| \le \varepsilon\, \alpha$. Since ε is arbitrary, it follows that $\lim_{n\to\infty} \gamma_n = 0$.

Remark. If both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent then their Cauchy product is also absolutely convergent.

Theorem (Abel). Assume that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two convergent series and their Cauchy product is also convergent. Then its sum is $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\, b_{n-k} = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$.

Remark. In general it is not true that the Cauchy-product of two convergent series is convergent.

For example let $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$. These are Leibniz series, so they are convergent.

Then $c_n = \sum_{k=0}^{n} a_k\, b_{n-k} = \sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\cdot\sqrt{n-k+1}}$.

Using the AM-GM inequality $\frac{a+b}{2} \geq \sqrt{a\,b}$, we get that

$| c_n | = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\cdot\sqrt{n-k+1}} \geq \sum_{k=0}^{n} \frac{2}{(k+1)+(n-k+1)} = \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2}{n+2}(n+1)$, since

the terms are independent of k.

Therefore $\left| c_n \right| \geq 2\cdot\frac{n+1}{n+2} \longrightarrow 2$, so $\lim_{n\to\infty} c_n \neq 0 \implies$ the Cauchy-product is divergent.

Examples

Example 1. If $| x | < 1$ then $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$ and

$\sum_{k=0}^{\infty} (-x)^k = 1 - x + x^2 - x^3 + \ldots = \frac{1}{1+x}$.

	1	x	x^2	x^3	...
1	1	x	x^2	x^3	
$-x$	$-x$	$-x^2$	$-x^3$		
x^2	x^2	x^3			
$-x^3$	$-x^3$				
...					

The Cauchy-product is $\sum_{n=0}^{\infty}\sum_{k=0}^{n} x^k(-x)^{n-k} = 1 + (x - x) + (x^2 - x^2 + x^2) + (x^3 - x^3 + x^3 - x^3) + \ldots =$

$= 1 + 0 + x^2 + 0 + x^4 + 0 + x^6 + \ldots = \sum_{k=0}^{\infty} x^{2k} = \sum_{k=0}^{\infty} (x^2)^k = \frac{1}{1-x^2} = \frac{1}{1-x}\cdot\frac{1}{1+x} = \left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{k=0}^{\infty} (-x)^k\right)$

Example 2. Since $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ if $| x | < 1$ then

$$\frac{1}{(1-x)^2} = \left(\sum_{k=0}^{\infty} x^k\right)^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} x^k\, x^{n-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} x^n = \sum_{n=0}^{\infty} (n+1)\, x^n$$

Example 3. $\left(\sum_{k=0}^{\infty} \frac{1}{n!}\right)^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} = \sum_{n=0}^{\infty} \frac{2^n}{n!}$