Calculus 1 - 09

Power series

Definitions. The series $\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots$ is called a **power series** with center x_0, where a_n is the coefficient of the nth term.

The domain of convergence of the power series is $H = \left\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges} \right\}$.

The **radius of convergence** of the power series is $R = \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$.

Remarks. H is not empty, since the series converges for $x = x_0$.

Since $\sqrt[n]{|a_n|} \geq 0$, then $0 \leq \limsup \sqrt[n]{|a_n|} \leq \infty$.

If $\limsup \sqrt[n]{|a_n|} = \infty$ then $R = 0$ and if $\limsup \sqrt[n]{|a_n|} = 0$ then $R = \infty$.

If $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$ exists then $R = \lim_{n\to\infty} \left| \dfrac{a_n}{a_{n+1}} \right|$.

Theorem (Cauchy-Hadamard): Denote by R the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. Then

(1) if $|x - x_0| < R$, then the series is absolutely convergent, and

(2) if $|x - x_0| > R$, then the series is divergent.

Proof. We define $\dfrac{1}{+0} = +\infty$ and $\dfrac{1}{+\infty} = 0$. By the root test

$$\limsup \sqrt[n]{|a_n| \cdot |x-x_0|^n} = \left| x - x_0 \right| \cdot \limsup \sqrt[n]{|a_n|} = \frac{|x-x_0|}{R}$$

Then $\dfrac{|x-x_0|}{R} < 1 \iff |x-x_0| < R \implies$ the series is absolutely convergent

and $\dfrac{|x-x_0|}{R} > 1 \iff |x-x_0| > R \implies$ the series is divergent.

Consequence. (1) If $R = 0$ then for all $x \neq x_0$, $|x-x_0| > 0 = R$, so the series diverges and if $x = x_0$ then it converges. Then $H = \{x_0\}$.

(2) If $R = \infty$ then for all $x \in \mathbb{R}$, $|x - x_0| < R$, so the series is absolutely convergent. Then $H = \mathbb{R}$.

(3) If $0 < R < \infty$, then $(x_0 - R, x_0 + R) \subset H \subset [x_0 - R, x_0 + R]$ and the endpoints of the interval must be investigated separately.

Power series, interval of convergence

Exercise 1: Find the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\ 2^n}\ (x-1)^n$$

Solution:

The coefficients are $a_n = \dfrac{(-1)^n}{n\ 2^n}$ and the base point is $x_0 = 1$. Applying the root test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{|(-1)^n|}{n\ 2^n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}\ 2} = \frac{1}{2} = \frac{1}{R} \quad \Longrightarrow \quad R = 2$$

We investigate the convergence at the endpoints:

If $x = 3$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n} \cdot 2^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ convergent by the Alternating Series Theorem (but not absolutely convergent)

If $x = -1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 2^n} \cdot (-2)^n = \sum_{n=1}^{\infty} \frac{1}{n}\ (-1)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n}$ divergent (the harmonic series)

The interval of convergence is: $(-1, 3]$.

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Exercise 2: Find the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} (-1)^n\ \frac{2n+1}{(2n)!}\ (x+7)^n\ , \qquad R = ?$$

Solution:

The coefficients are $a_n = (-1)^n\ \dfrac{2n+1}{(2n)!}$ and the base point is $x_0 = -7$. Applying the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+3)\ (2n)!}{(2n+2)!\ (2n+1)} = \lim_{n\to\infty} \frac{2n+3}{2n+1}\ \frac{1}{(2n+2)(2n+1)} = 0 = \frac{1}{R} \quad \Longrightarrow \quad R = \infty$$

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Exercise 3: Find the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(n+2)^{n^2}}{(n+6)^{n^2+1}}\ x^n\ , \qquad R = ?$$

Solution:

The coefficients are $a_n = \dfrac{(n+2)^{n^2}}{(n+6)^{n^2+1}}$ and the base point is $x_0 = 0$. Applying the root test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n+2}{n+6}\right)^n \frac{1}{\sqrt[n]{n+6}} = \lim_{n\to\infty} \frac{\left(1+\dfrac{2}{n}\right)^n}{\left(1+\dfrac{6}{n}\right)^n}\ \frac{1}{\sqrt[n]{n+6}} = \frac{e^2}{e^6} \cdot 1 = \frac{1}{e^4} = \frac{1}{R}$$

$$\Longrightarrow \quad R = e^4$$

Here we used that $1 < \sqrt[n]{n+6} < \sqrt[n]{7}\ \sqrt[n]{n}$ and thus $\sqrt[n]{n+6} \to 1$ by the Sandwich Theorem.

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Exercise 4: Find the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n!}\ x^n\ , \qquad R = ?$$

Solution:

The coefficients are $a_n = \dfrac{(n+1)^n}{n!}$ and the base point is $x_0 = 0$. Applying the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+2)^{n+1}\ n!}{(n+1)!\,(n+1)^n} = \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^{n+1} = \lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n+1} = \mathrm{e} = \frac{1}{R}$$

$$\Longrightarrow \quad R = \frac{1}{\mathrm{e}}$$

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Exercise 5: Find the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(-2)^n\ (n+3)}{n^2+3}\ x^n$$

Solution:

The coefficients are $a_n = \dfrac{(-2)^n\ (n+3)}{n^2+3}$ and the base point is $x_0 = 0$. Applying the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-2)^{n+1}\ (n+4)}{(n+1)^2+3}\right| \cdot \left|\frac{n^2+3}{(-2)^n\ (n+3)}\right| = \lim_{n\to\infty} 2 \cdot \frac{n+4}{n+3} \cdot \frac{n^2+3}{n^2+2n+4} =$$

$$= 2\cdot 1\cdot 1 = 2 = \frac{1}{R} \ \Longrightarrow R = \frac{1}{2}$$

The endpoints:

If $x = -\dfrac{1}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{n+3}{n^2+3}$. Since $\dfrac{n+3}{n^2+3} \geq \dfrac{n+0}{n^2+3n^2} = \dfrac{1}{4n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n}$ is divergent then $\displaystyle\sum_{n=1}^{\infty} \frac{n+3}{n^2+3}$ is also divergent by the Comparison Test.

If $x = \dfrac{1}{2}$: $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+3}{n^2+3}$ is convergent by the Alternating Series Theorem.

The interval of convergence is: $\left(-\dfrac{1}{2}\,,\,\dfrac{1}{2}\right]$.

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Exercise 6: Find the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(2x+4)^n}{n^2\ 3^n}$$

Solution:

The series can be written as $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2\ 3^n}\ (x+2)^n$, so the coefficients are $a_n = \dfrac{2^n}{n^2\ 3^n}$ and the base point is $x_0 = -2$. Applying the root test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2\ 3^n}} = \lim_{n\to\infty} \frac{2}{3\ (\sqrt[n]{n})^2} = \frac{2}{3} = \frac{1}{R} \quad \Longrightarrow \quad R = \frac{3}{2}$$

The endpoints:

If $x = -\dfrac{7}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is convergent (by the Alternating Series Theorem, or: it is absolutely convergent)

If $x = -\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

The interval of convergence is: $\left[-\frac{7}{2}, -\frac{1}{2}\right]$.

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Exercise 7: Find the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{n}{2^n}\, x^{3n} = \frac{1}{2}\, x^3 + \frac{2}{2^2}\, x^6 + \frac{3}{2^3}\, x^9 + \frac{4}{2^4}\, x^{12} + \cdots , \qquad R = ?$$

1st solution:

The coefficients are $\quad a_n = \begin{cases} 0\,, & \text{if n is not divisible by 3} \\ \dfrac{n/3}{2^{n/3}}\,, & \text{if n is divisible by 3} \end{cases}$

Then $\quad \sqrt[n]{|a_n|} = \begin{cases} 0\,, & \text{if n is not divisible by 3} \\ \sqrt[n]{\dfrac{n/3}{2^{n/3}}} = \dfrac{\sqrt[n]{n}}{\sqrt[n]{3}\ \sqrt[3]{2}}\,, & \text{if n is divisible by 3} \end{cases}$

$\Longrightarrow$ The accumulation points are: $t_1 = 0\,,\ t_2 = \frac{1}{\sqrt[3]{2}}$

$\Longrightarrow \quad \overline{\lim} \sqrt[n]{|a_n|} = \frac{1}{\sqrt[3]{2}} = \frac{1}{R} \quad \Longrightarrow \quad R = \sqrt[3]{2}$

2nd solution:

By the substitution $y = x^3$ the series can be written in the form

$$\sum_{n=1}^{\infty} b_n\, y^n := \sum_{n=1}^{\infty} \frac{n}{2^n}\, y^n$$

The coefficients are $b_n = \frac{n}{2^n}$ and the base point is $y_0 = 0$. Applying the root test:

$$\lim_{n\to\infty} \sqrt[n]{|b_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{2^n}} = \frac{1}{2} = \frac{1}{R_y} \quad \Longrightarrow \quad R_y = 2$$

The radius of convergence of the original series can be determined in the following way:

$$|y| < 2 \quad \Longrightarrow \quad \left|x^3\right| = |x|^3 < 2 \quad \Longrightarrow \quad |x| < \sqrt[3]{2} \quad \Longrightarrow \quad R = \sqrt[3]{2}$$

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Exercise 8: Find the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{n+1}{9^n}\, (x-2)^{2n}$$

Solution:

By the substitution $y := (x-2)^2$ the series can be written in the form: $\sum_{n=1}^{\infty} \frac{n+1}{9^n}\, y^n$

The coefficients are $a_n = \frac{n+1}{9^n}$ and the base point is $y_0 = 0$. Applying the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+2)\ 9^n}{9^{n+1}\ (n+1)} = \lim_{n\to\infty} \frac{1}{9} \cdot \frac{n+2}{n+1} = \frac{1}{9} \quad \Longrightarrow \quad R_1 = 9$$

The radius of convergence of the original series can be determined in the following way:

$$|y| < 9 \quad \Longrightarrow \quad |(x-2)^2| < 9 \quad \Longrightarrow \quad \underbrace{|x-2| < 3}_{-1<x<5} \quad \Longrightarrow \quad R = 3$$

The endpoints:

If $x=-1$ or $x=5$ then $\sum_{n=1}^{\infty} \frac{n+1}{9^n}\ (-1-2)^{2n} = \sum_{n=1}^{\infty} \frac{n+1}{9^n}\ (5-2)^{2n} = \sum_{n=1}^{\infty} (n+1)$.

This series is divergent by the nth term test, so the interval od convergence is $(-1,5)$.

Remark: The endpoints can be investigated in both the original and the new series.

Practice exercises

Exercise 9: Find the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{n^3}\ (x+1)^n$$

Exercise 10: Find the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{n\sqrt{n}}{2^{2n}}\ (4-2x)^n$$

Exercise 11: Find the interval of convergence of the following power series:

$$a)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\ 3^{2n}}\ x^n \qquad b)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\ 3^{2n}}\ (x+2)^{2n}$$

Exercise 12: Find the interval of convergence of the following power series:

$$a)\ \sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt[3]{n}}\ x^n \qquad b)\ \sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt[3]{n}}\ x^{2n}$$

Exercise 13: Find the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n\ (2x)^n}{\sqrt{n}\ 5^n}$$

Results

Exercise 9: $\left[-\frac{5}{4}, -\frac{3}{4}\right]$. **Exercise 10:** $(0,4)$. **Exercise 11: a)** $[-9,9]$ **b)** $[-5,1]$
Exercise 12: a) $\left(-\frac{1}{3}, \frac{1}{3}\right]$ **b)** $\left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$ **Exercise 13:** $\left(-\frac{5}{2}, \frac{5}{2}\right]$