Calculus 1 - 15

Taylor polynomial, Taylor series

Taylor polynomial

Definition. Let f be at least n times differentiable at x_0. Then the polynomial

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is the ***n*th Taylor polynomial of *f* at x_0**. (If $x_0 = 0$: Maclaurin polynomial.)

Example. Let $f(x) = \sin x$ and $x_0 = 0$.

$f(x) = \sin x \quad \Longrightarrow f(0) = 0 \quad \Longrightarrow T_0(x) = 0$

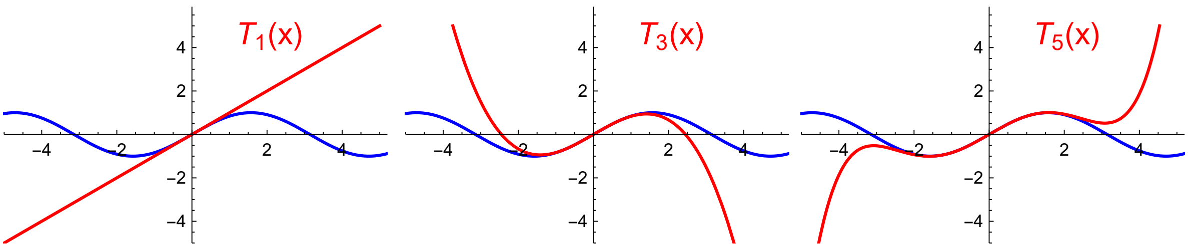
$f'(x) = \cos x \quad f'(0) = 1 \quad T_1(x) = 0 + 1 \cdot x = x$

$f''(x) = -\sin x \quad f''(0) = 0 \quad T_2(x) = 0 + 1 \cdot x + \frac{0}{2!} x^2 = x$

$f'''(x) = -\cos x \quad f'''(0) = -1 \quad T_3(x) = 0 + 1 \cdot x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 = x - \frac{x^3}{6}$

$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0 \quad T_4(x) = 0 + 1 \cdot x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 + \frac{0}{4!} x^4 = x - \frac{x^3}{6}$

$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1 \quad T_5(x) = 0 + 1 \cdot x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5 = x - \frac{x^3}{6} + \frac{x^5}{120}$



Theorem (Uniqueness of the Taylor polynomial).
Assume that f is at least n times differentiable at x_0. Then $T_n(x)$ is the unique polynomial of degree at most n for which $T_n^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0, 1, \ldots, n$.

Proof. Express $T_n(x)$ as powers of $(x - x_0)$:

$$T_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + a_4(x - x_0)^4 + \ldots + a_n(x - x_0)^n$$
$$\Longrightarrow T_n(x_0) = a_0 = f(x_0)$$
$$T_n'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \ldots + n a_n(x - x_0)^{n-1}$$
$$\Longrightarrow T_n'(x_0) = a_1 = f'(x_0)$$
$$T_n''(x) = 2a_2 + 3 \cdot 2 \cdot a_3(x - x_0) + 4 \cdot 3 \cdot a_4(x - x_0)^2 + \ldots + n(n-1)a_n(x - x_0)^{n-2}$$
$$\Longrightarrow T_n''(x_0) = 2a_2 = f''(x_0)$$
$$T_n'''(x) = 3! \cdot a_3 + \ldots + n(n-1)(n-2)a_n(x - x_0)^{n-3}$$
$$\Longrightarrow T_n'''(x_0) = 3!\, a_3 = f'''(x_0)$$

Repeating the differentiation, we get that

$$T_n^{(k)}(x_0) = k!\, a_k = f^{(k)}(x_0),\ \ldots,\ \ T_n^{(n)}(x_0) = n!\, a_n = f^{(n)}(x_0) \implies a_k = \frac{f^{(k)}(x_0)}{k!},\quad k = 0, 1, \ldots, n.$$

Theorem. If f is n times differentiable at x_0 then $\lim\limits_{x\to x_0} \dfrac{f(x) - T_n(x)}{(x - x_0)^n} = 0.$

Proof. $$\lim_{x\to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} \overset{L'H}{=} \lim_{x\to x_0} \frac{f'(x) - T_n'(x)}{n(x - x_0)^{n-1}} \overset{L'H}{=} \lim_{x\to x_0} \frac{f''(x) - T_n''(x)}{n(n-1)(x - x_0)^{n-2}} \overset{L'H}{=} \ldots \overset{L'H}{=} \lim_{x\to x_0} \frac{f^{(n)}(x) - T_n^{(n)}(x)}{n!\,(x - x_0)^{n-n}} = \frac{0}{n!} = 0$$

Except the last one, the fractions are of the form $\frac{0}{0}$ so the L'Hospital's rule can be applied.

Taylor's theorem

Theorem (Taylor's theorem). Assume that f is at least $(n+1)$ times differentiable on the interval $(x_0 - \delta,\ x_0 + \delta)$ and $x \in (x_0 - \delta,\ x_0 + \delta)$. Then there exists a number ξ between x and x_0 (that is, $x_0 < \xi < x$ or $x < \xi < x_0$) such that

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

This expression is called the **Lagrange form of the remainder term.**

Proof. Assume that $x_0 < x$ and $y \in [x_0, x]$ (x is fixed).

Define the following function:

$$F(y) = f(y) + f'(y)(x - y) + \frac{f''(y)}{2!}(x - y)^2 + \ldots + \frac{f^{(n)}(y)}{n!}(x - y)^n$$

Differentiating F with respect to y, we get a telescoping sum:

$$F'(y) = \mathbf{f'(y)} + (\mathbf{f''(y)(x - y)} - \mathbf{f'(y)}) + \left(\frac{f'''(y)}{2!}(x - y)^2 - \frac{\mathbf{f''(y)}}{\mathbf{2!}}\cdot\mathbf{2}\cdot(\mathbf{x - y})\right) + \ldots +$$
$$+\left(\frac{f^{(n+1)}(y)}{n!}(x - y)^n - \frac{f^{(n)}(y)}{n!}\cdot n\cdot(x - y)^{n-1}\right) = \frac{f^{(n+1)}(y)}{n!}(x - y)^n$$

$\implies F(x) = f(x)$

$F(x_0) = T_n(x_0)$

$F'(y) = \dfrac{f^{(n+1)}(y)}{n!}(x - y)^n$

Let $G(y) = (x - y)^{n+1} \implies G(x) = 0$

$G(x_0) = (x - x_0)^{n+1}$

$G'(y) = -(n+1)(x - y)^n$

Both F and G are continuous on $[x_0, x]$, differentiable on (x_0, x) and $G'(y) \neq 0\ \forall\, y \in (x_0, x)$, so by Cauchy's intermediate value theorem there exists $\xi \in (x_0, x)$ such that

$$\frac{f(x) - T_n(x_0)}{0 - (x - x_0)^{n+1}} = \frac{F(x) - F(x_0)}{G(x) - G(x_0)} = \frac{F'(\xi)}{G'(\xi)} = \frac{\frac{f^{(n+1)}(\xi)}{n!}(x - \xi)^n}{-(n+1)(x - \xi)^n} = -\frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$\implies f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

Remark. If $n = 0$ then we obtain Lagrange's mean value theorem:
$$f(x) = T_0(x) + R_0(x) = f(x_0) + f'(\xi)(x - x_0)$$

Taylor series

Definition. If f is infinitely many times differentiable at x_0 then the power series
$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$ is called the **Taylor series** of f at x_0.

Definition. The functions $g_0, g_1, \ldots, g_n, \ldots$ are **uniformly bounded** on the set $D = \bigcap D_{g_n}$ if
$$\exists K \in \mathbb{R}: \quad |g_n(x)| \le K \quad \text{if } x \in D,\ n \in \mathbb{N},$$
that is, the functions have a common bound K on the set D.

Example. The functions $\sin(x)$, $2\sin(2x)$, $3\sin(3x)$, ..., $n\sin(nx)$, ... are each bounded but not uniformly bounded altogether.
The functions $\sin(x)$, $\sin(2x)$, $\sin(3x)$, ..., $\sin(nx)$, ... are uniformly bounded, $K = 1$ is a suitable common bound.

Theorem. If f is infinitely many times differentiable on $(x_0 - R, x_0 + R)$ (R is the radius of convergence of T) and the functions $f, f', f'', f''', \ldots, f^{(n)}, \ldots$ are uniformly bounded on this interval then $f(x) = T(x)$ for all $x \in (x_0 - R, x_0 + R)$.

Proof. We have seen that $f(x) = T_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$.

Using uniform boundedness and the limit $\lim_{n\to\infty} \frac{a^n}{n!} = 0$, we can give an upper estimation for the remainder term:
$$|f(x) - T_n(x)| = |R_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \right| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!} |x - x_0|^{n+1} \le K \cdot \frac{R^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0$$
Therefore $T_n(x) \longrightarrow f(x)$ and thus $f(x) = T(x)$.

Remark. Using this theorem, it can be shown that many elementary functions are equal to their power series.

Theorem. (1) $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots,\quad x \in \mathbb{R}$

(2) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots,\quad x \in \mathbb{R}$

(3) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots,\quad x \in \mathbb{R}$

(4) $\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots,\quad x \in \mathbb{R}$

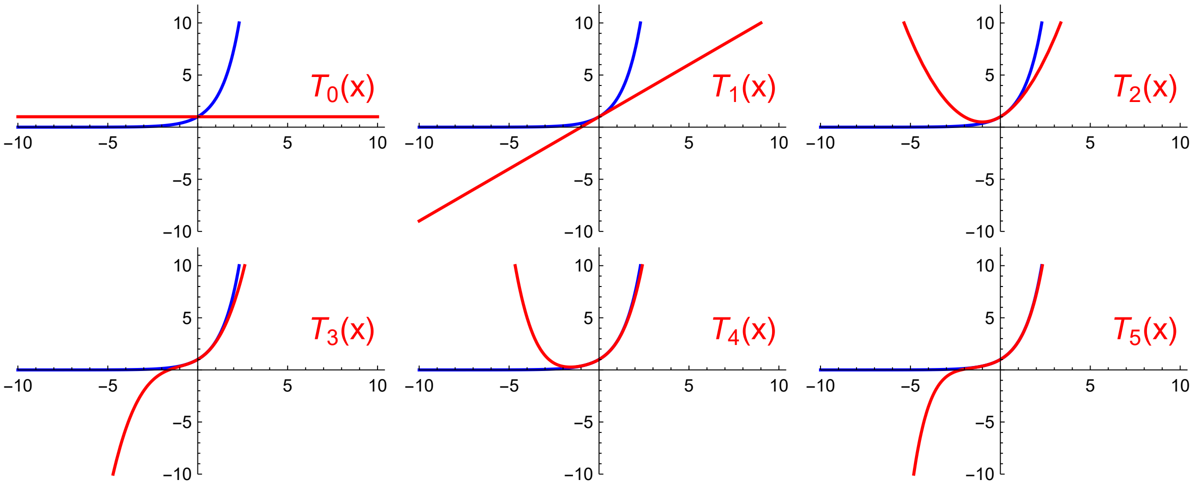
(5) $\cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots,\quad x \in \mathbb{R}$

In each case, the interval of convergence is the set of real numbers.

Proof. (1) Let $f(x) = e^x$ and $x_0 = 0 \implies f^{(k)}(x) = e^x \ \ \forall k \in \mathbb{N} \implies f^{(k)}(0) = 1$.

By the ratio test the radius of convergence is $\lim\limits_{n\to\infty} \dfrac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim\limits_{n\to\infty} \dfrac{(n+1)!}{n!} = \lim\limits_{n\to\infty} (n+1) = \infty$

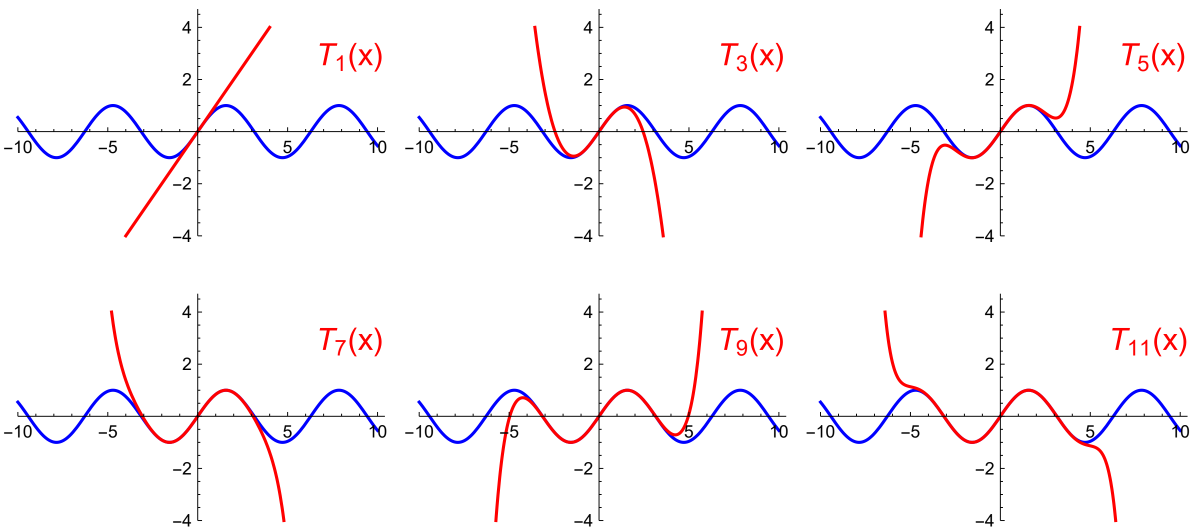
$\implies$ the interval of convergence is $\mathbb{R}$.
The derivatives are not bounded on $\mathbb{R}$, however, they are uniformly bounded on any closed interval $[a, b]$ and $\left| f^{(k)}(x) \right| \leq e^b = K \implies f(x) = T(x)$ if $x \in [a, b]$.
Since this equality holds for **any** closed interval $[a, b]$ then $f(x) = T(x) \ \ \forall x \in \mathbb{R}$ also holds.



(2) Let $f(x) = \sin x$ and $x_0 = 0$.
$f(x) = \sin x, \ \ f'(x) = \cos x, \ \ f''(x) = -\sin x, \ \ f'''(x) = -\cos x, \ \ f^{(4)}(x) = f(x) = \sin x$
$f(0) = 0, \qquad f'(0) = 1, \qquad f''(0) = 0, \qquad f'''(0) = -1, \qquad f^{(4)}(0) = f(0) = 0$

From here it is repeated periodically. The derivatives are uniformly bounded on $\mathbb{R}$

$$\implies \cos x = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots + \frac{f^{(k)}(0)}{k!} x^k + \ldots =$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$



(3) Let $f(x) = \cos x$ and $x_0 = 0$.

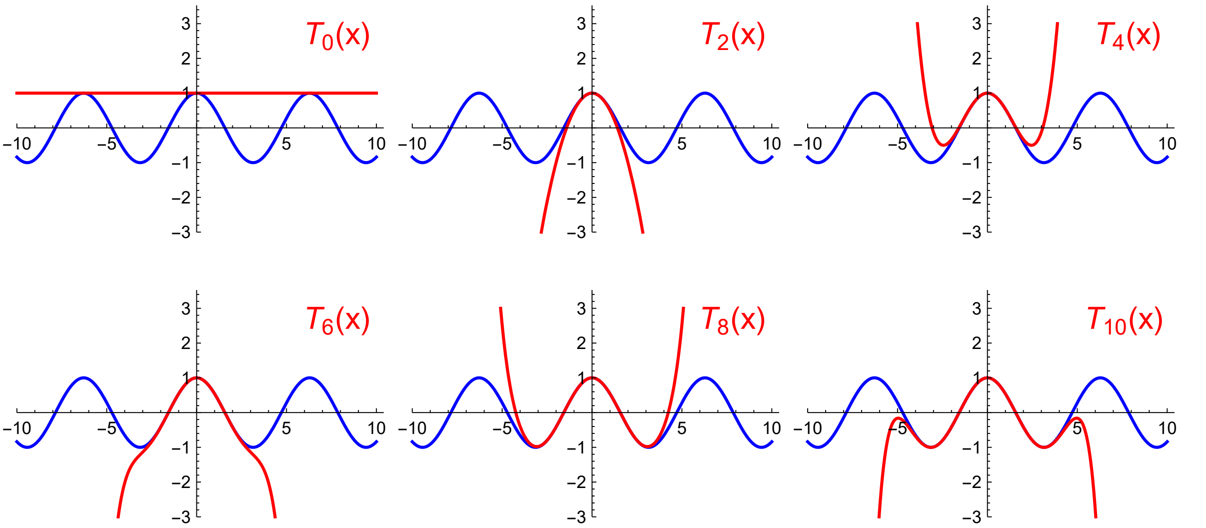
$f(x) = \cos x,\quad f'(x) = -\sin x,\quad f''(x) = -\cos x,\quad f'''(x) = \sin x,\quad f^{(4)}(x) = f(x) = \cos x$

$f(0) = 1,\quad f'(0) = 0,\quad f''(0) = -1,\quad f'''(0) = 0,\quad f^{(4)}(0) = f(0) = 1$

From here it is repeated periodically. The derivatives are uniformly bounded on $\mathbb{R}$

$$\Longrightarrow \cos x = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots + \frac{f^{(k)}(0)}{k!} x^k + \ldots =$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$



(4), (5)

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \ (x \in \mathbb{R}) \quad \Longrightarrow \quad \sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n - (-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \ (x \in \mathbb{R}) \qquad \cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n + (-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$

Remark. The Taylor series of $f(x) = e^x$ at x_0 is $e^x = e^{x_0} e^{x - x_0} = \sum_{n=0}^{\infty} \frac{e^{x_0}}{n!} (x - x_0)^n$

Examples. (1) The Taylor series of $f(x) = \sin x \cdot \cos x$ at $x_0 = 0$ is

$$f(x) = \sin x \cdot \cos x = \frac{1}{2} \sin 2x = \frac{1}{2} \cdot \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots\right) \ (x \in \mathbb{R})$$

(2) The Taylor series of $f(x) = e^{-x^2}$ at $x_0 = 0$ is $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots \ (x \in \mathbb{R})$

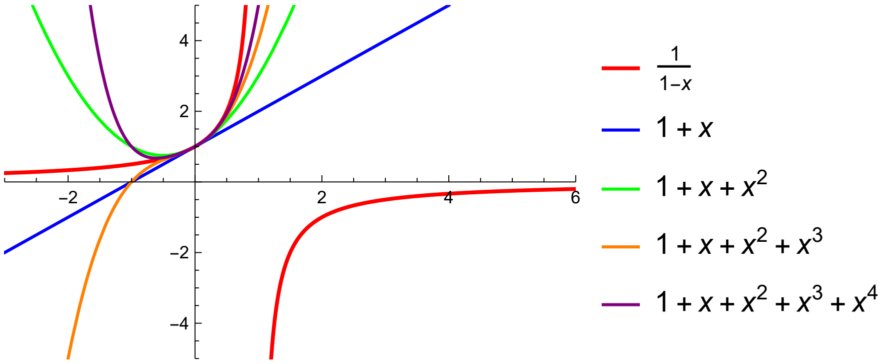
Example. It is known that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x| < 1$ (sum of a geometric series).

Find the Taylor series of $f(x) = \frac{1}{1-x}$ at $x_0 = 0$.

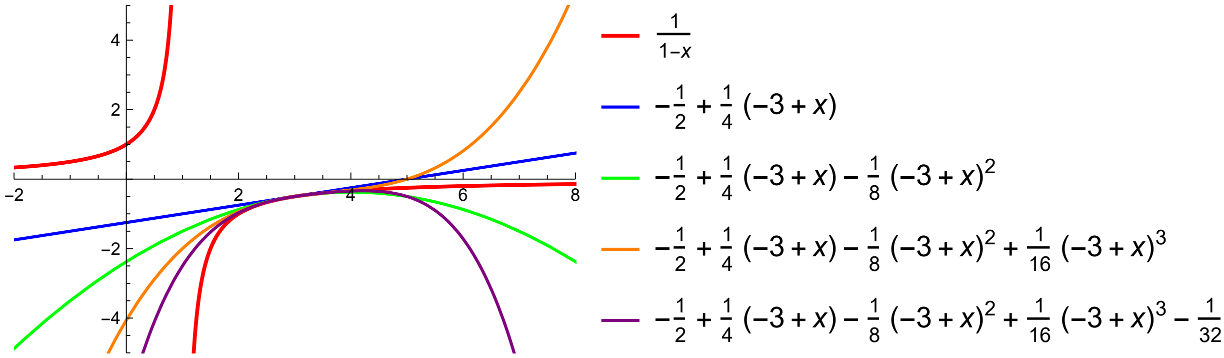
$$f'(x) = \frac{1}{(1-x)^2},\quad f''(x) = \frac{2}{(1-x)^3},\quad f'''(x) = \frac{3!}{(1-x)^4},\quad \ldots,\quad f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \Longrightarrow f^{(n)}(0) = n!$$

$$\Longrightarrow T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n \quad \Longrightarrow \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ \text{ if } |x| < 1.$$

Taylor polynomials of $f(x) = \dfrac{1}{1-x}$ with center $x_0 = 0$:



Taylor polynomials of $f(x) = \dfrac{1}{1-x}$ with center $x_0 = 3$:



Term by term differentiation and integration

Theorem. If $|x - x_0| < R$ where R is the radius of convergence of the

Taylor series $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then $f'(x) = \sum_{n=0}^{\infty} n\, a_n (x - x_0)^{n-1}$

and if $f'(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then $f(x) = f(x_0) + \sum_{n=0}^{\infty} a_n \dfrac{(x - x_0)^{n+1}}{n+1}$.

Example. $(e^x)' = \left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \ldots\right)' = 0 + 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots = e^x$

$$(\sin x)' = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots\right)' = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots = \cos x$$

Statement. $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} + \ldots = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ if $x \in [-1, 1]$.

Proof. $f(x) = \arctan x \implies f'(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \ldots = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$f'(x)$ is the sum of a geometric series with ratio $q = -x^2$.

It is convergent $\iff |q| = |-x^2| = |x|^2 < 1 \implies R = 1$

The interval of convergent for f' is $(-1, 1)$.

$$\implies f(x) = \arctan x = f(0) + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \ldots$$

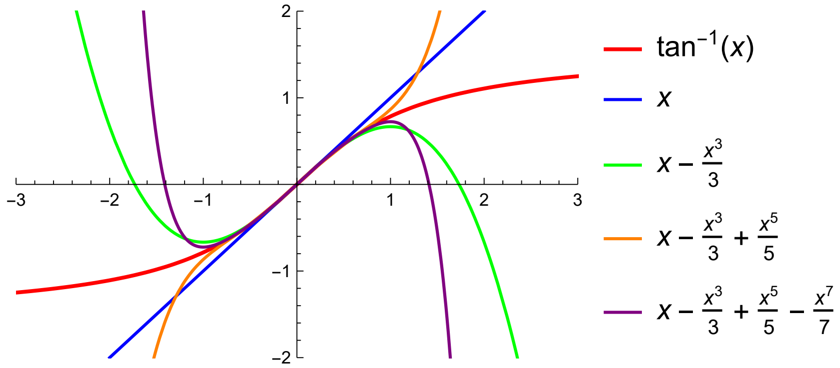
The radius of convergence doesn't change but the endpoints of the interval of convergence can change. Here both endpoints change.

If $x = 1$ then $f(1) = \text{arctg}\, 1 = \frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots,$

this is a Leibniz type series, so it is convergent.

If $x = -1$ then $f(-1) = -f(1)$, so the series is also convergent.

The interval of convergence for f is $[-1, 1]$.



In[5]:=
```
Series[Log[x], {x, 0, 10}]
```

Out[5]= $\text{Log}[x] + O[x]^{11}$

Statement. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ if $x \in (-1, 1]$.

Proof. $f(x) = \ln(1+x) \implies f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

$f'(x)$ is the sum a geometric series with ratio $q = -x$.

It is convergent $\iff |q| = |-x| = |x| < 1 \implies R = 1$.

The interval of convergent for f' is $(-1, 1)$.

$\implies f(x) = \ln(1+x) = f(0) + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \ldots$

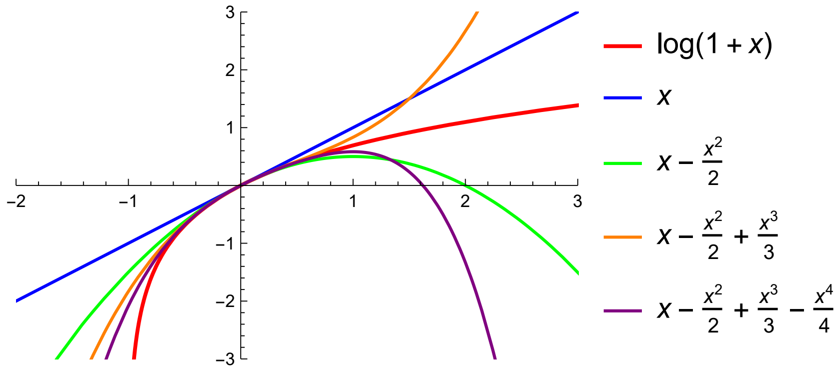
The radius of convergence doesn't change but the endpoints of the interval of convergence can change. Here the right endpoint changes.

If $x = -1$ then $f(x)$ is not defined. If $x = 1$ then

$f(1) = \ln 2 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \ldots$, this is a Leibniz type series,

so it is convergent.

The interval of convergence for f is: $(-1, 1]$.



Binomial series

Example. Find the kth Taylor polynomial of the function $f(x)=(1+x)^{\alpha}$ (where $\alpha\in\mathbb{R}$) about $x_0=0$.

$$f(x)=(1+x)^{\alpha} \quad\Longrightarrow\quad f(0)=1$$
$$f'(x)=\alpha(1+x)^{\alpha-1} \quad\Longrightarrow\quad f'(0)=\alpha$$
$$f''(x)=\alpha(\alpha-1)(1+x)^{\alpha-2} \quad\Longrightarrow\quad f''(0)=\alpha(\alpha-1)$$
$$\ldots$$
$$f^{(k)}(x)=\alpha(\alpha-1)(\alpha-k+1)(1+x)^{\alpha-k} \Longrightarrow f^{(k)}(0)=\alpha(\alpha-1)\ldots(\alpha-k+1)$$

$$\Longrightarrow T_k(x)=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3\ldots+\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}x^k.$$

Remark. If $\alpha=n\in\mathbb{N}$ then by the binomial theorem

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k \text{ where } \binom{n}{k}=\frac{n!}{k!\,(n-k)!}=\frac{n(n-1)\ldots(n-k+1)}{k!}.$$

Definition (Generalized binomial coefficient). If $\alpha\in\mathbb{R}$ and $k\in\mathbb{R}$ then

$$\binom{\alpha}{k}:=\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} \text{ and } \binom{\alpha}{0}:=1$$

Remark. The Taylor series of $f(x)=(1+x)^{\alpha}$ (where $\alpha\in\mathbb{R}$) is $T(x)=\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$.

It is called a **binomial series**.

Theorem. The radius of convergence of the binomial series $\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$ is $R=1$.

Proof. $a_k=\binom{\alpha}{k}=\dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$

$$a_{k+1}=\binom{\alpha}{k+1}=\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)(\alpha-k)}{(k+1)!}=\binom{\alpha}{k}\frac{\alpha-k}{k+1}$$

By the ratio test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\left|\frac{\alpha-k}{k+1}\right|=1=\frac{1}{R}\Longrightarrow R=1$$

Theorem (Sum of a binomial series).

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \quad \text{for all } x \in (-1, 1), \quad \alpha \in \mathbb{R}.$$

Proof. The derivatives of $f(x)$ are not uniformly bounded, so we prove the equality of

$f(x) = (1+x)^\alpha$ and $T(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ in the following way.

We show that for all $x \in (-1, 1)$: $\left(\frac{T(x)}{f(x)}\right)' \equiv 0 \implies \frac{T(x)}{f(x)} \equiv$ constant.

Since $f(0) = T(0) = 1$ then $\frac{T(x)}{f(x)} = \frac{T(0)}{f(0)} = 1 \implies T(x) = f(x)$ if $x \in (-1, 1)$.

The derivative is

$$\left(\frac{T(x)}{f(x)}\right)' = \frac{T' \cdot f - T \cdot f'}{f^2} = \frac{T' \cdot (1+x)^\alpha - T \cdot \alpha (1+x)^{\alpha-1}}{f^2} = \frac{(1+x)^{\alpha-1}}{(1+x)^{2\alpha}} ((1+x)\, T' - \alpha \cdot T)$$

$\implies$ it is enough to show that $(1+x)\, T' - \alpha \cdot T \equiv 0$.
For this we use the power series of T and T'.

$$T(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \ldots + \binom{\alpha}{k} x^k + \ldots$$

$$\alpha\, T(x) = \alpha + \alpha^2 x + \frac{\alpha^2(\alpha-1)}{2!} x^2 + \ldots + \alpha \binom{\alpha}{k} x^k + \ldots$$

$$T'(x) = \alpha + \frac{\alpha(\alpha-1)}{1!} x + \frac{\alpha(\alpha-1)(\alpha-2)}{2!} x^2 + \ldots + k \binom{\alpha}{k} x^{k-1} + (k+1) \binom{\alpha}{k+1} x^k + \ldots$$

$$x\, T'(x) = \alpha x + \frac{\alpha(\alpha-1)}{1!} x^2 + + \ldots + k \binom{\alpha}{k} x^k + \ldots$$

The above expression as a power series is

$$(1+x)\, T' - \alpha \cdot T = \sum_{k=0}^{\infty} \left((k+1) \binom{\alpha}{k+1} + k \binom{\alpha}{k} - \alpha \binom{\alpha}{k} \right) x^k$$

where the coefficient of x^k for all $k \in \mathbb{N}$ is

$$(k+1) \binom{\alpha}{k+1} + k \binom{\alpha}{k} - \alpha \binom{\alpha}{k} = \binom{\alpha}{k} \left((k+1) \frac{\alpha - k}{k+1} + k - \alpha \right) = 0.$$

Thus we proved that for all $x \in (-1, 1)$

$$(1+x)\, T'(x) - \alpha \cdot T(x) \equiv 0 \implies \left(\frac{T}{f}\right)' \equiv 0 \implies \frac{T(x)}{f(x)} = \frac{T(0)}{f(0)} = 1$$

$$\implies T(x) = f(x) \text{ if } x \in (-1, 1).$$

Example. If $f(x) = \arcsin x$, then $f'(x) = \frac{1}{\sqrt{1-x^2}} = (1 + (-x^2))^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k x^{2k}$ if $|x| < 1$

$$\implies \arcsin x = \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{(-1)^k x^{2k+1}}{2k+1} \quad \text{if } |x| < 1.$$

Remark. Taylor's formula $\implies$

$(1+x)^\alpha = T_n(x) + R_n(x)$, where

$$T_n(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k \quad \text{and} \quad R_n(x) = \binom{\alpha}{n+1} (1+\xi)^{\alpha-n-1} x^{n+1}, \text{ where } 0 < \xi < x \text{ or } x < \xi < 0.$$

Exercises

Exercise 1. Estimate the value of $\sqrt{2}$ by the Taylor polynomial of order 2 of $f(x) = \sqrt{1+x}$ at center $x_0 = 0$. Give an upper bound for the error for the error of the approximation.

Solution. The derivatives and the substitution values:

$$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}, \quad f(0) = 1$$

$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1+x}}, \quad f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} = -\frac{1}{4(1+x)^{3/2}}, \quad f''(0) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} = \frac{3}{8(1+x)^{5/2}}$$

The Taylor polynomial of order 2:

$$T_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = 1 + \frac{1}{2}x - \frac{1}{4\cdot 2!}x^2$$

$$f(x) \approx T_2(x), \text{ that is, } \sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{4\cdot 2!}x^2$$

$$\text{If } x = 1 \text{ then } \sqrt{2} \approx T_2(1) = 1 + \frac{1}{2} - \frac{1}{4\cdot 2!} = 1.375$$

Lagrange remainder term: $R_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-x_0)^3$, where $x_0 = 0$, $x = 1$, $0 < \xi < 1$

The value of ξ is not known so we can only estimate the error.

$$|E| = |R_2(x)| = \left| \frac{3}{8(1+\xi)^{5/2}} \cdot \frac{1}{3!}(1-0)^3 \right| = \frac{1}{16(1+\xi)^{5/2}} < \frac{1}{16(1+0)^{5/2}} = \frac{1}{16} = 0.0625$$

Remark: $\sqrt{2} \approx 1.414213562\ldots$

The approximation is $\sqrt{2} \approx T_2(1) = 1.375$

$1.4142 - 1.375 = 0.0392$

Remark: $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$

$$\Longrightarrow T_2(x) = \sum_{k=0}^{2}\binom{1/2}{k}x^k = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2!}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$R_2(x) = \binom{1/2}{3}(1+\xi)^{\frac{1}{2}-3}x^3 = \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{3!}\cdot\frac{1}{(1+\xi)^{\frac{5}{2}}}\cdot x^3 = \frac{3}{8\cdot 3!}\cdot\frac{1}{(1+\xi)^{\frac{5}{2}}}\cdot x^3$$

Exercise 2. We estimate the value of ln(1.1) by the Taylor polynomial of order n of $f(x) = \ln(1+x)$ at center $x_0 = 0$. Find n if the error for the approximation is less than 10^{-8}.

Solution. The first few derivatives are

$$f(x) = \ln(1+x), \quad f(0) = 0 \qquad f'''(x) = \frac{2}{(1+x)^3}, \quad f'''(0) = 2 = 2!$$

$$f'(x) = \frac{1}{1+x}, \quad f'(0) = 1 = 0! \qquad f^{(4)}(x) = -\frac{6}{(1+x)^4}, \quad f^{(4)}(0) = -6 = -3!$$

$$f''(x) = -\frac{1}{(1+x)^2}, \quad f''(0) = -1 = -1! \qquad f^{(5)}(x) = \frac{24}{(1+x)^5}, \quad f^{(5)}(0) = 24 = 4! \text{ etc.}$$

$$\Longrightarrow \mathbf{f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}} \quad \Longrightarrow \quad f^{(n)}(0) = (-1)^{n-1}\cdot(n-1)!$$

The Taylor polynomial at $x_0 = 0$:

$$T_n(x) = 0 + 1\cdot(x-0) - \frac{1}{2!}(x-0)^2 + \frac{2}{3!}(x-0)^3 + \frac{-6}{4!}(x-0)^4 + \ldots = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 +$$

If $x = 0.1$: $\ln(1.1) \approx 0.1 - \frac{1}{2}0.1^2 + \frac{1}{3}0.1^3 - \frac{1}{4}0.1^4 + \ldots$

Lagrange remainder term: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$, where $x_0 = 0$, $x = 0.1$, $0 < \xi < 0.1$.

Taylor's theorem: $f(x) = T_n(x) + R_n(x)$

The error: $|E| = |f(x) - T_n(x)| = |R_n(x)| = \left|\frac{(-1)^n\cdot n!}{(1+\xi)^{n+1}\cdot(n+1)!}(x-x_0)^{n+1}\right| =$

$$= \left|\frac{(-1)^n\cdot n!}{(1+\xi)^{n+1}\cdot(n+1)!}(0.1-0)^{n+1}\right| = \frac{1}{(n+1)(1+\xi)^{n+1}}\cdot 0.1^{n+1} < \frac{1}{(n+1)\,\mathbf{(1+0)^{n+1}}}\cdot 0.1^{n+1} =$$

$$= \frac{0.1^{n+1}}{(n+1)} < 10^{-8} \Longrightarrow n \geq 7$$

Comparison of the numerical values:

$\ln(1.1) \approx 0.09531017980\ldots$

The approximation is

$$\ln(1.1) \approx T_7(0.1) = 0.1 - \frac{1}{2}\cdot 0.1^2 + \frac{1}{3}\cdot 0.1^3 - \frac{1}{4}\cdot 0.1^4 + \frac{1}{5}\cdot 0.1^5 - \frac{1}{6}\cdot 0.1^6 + \frac{1}{7}\cdot 0.1^7 \approx 0.09531018095\ldots$$

$\Longrightarrow$ the first 7 digits are accurate.