Calculus 1 - 16

Indefinite integrals

Antiderivative

Definition. If F is differentiable on the interval I and $F'(x) = f(x)$ for all $x \in I$
F is an **antiderivative** or **primitive function** of f.

Example. If $f(x) = \sin x \cos x = \frac{1}{2} \sin 2x$ then

$F(x) = \frac{\sin^2 x}{2}$ and $G(x) = -\frac{\cos 2x}{4}$ are primitive functions of f on $\mathbb{R}$
since $F'(x) = G'(x) = f(x)$.

Theorem. If F and G are antiderivatives of f on the interval I then there exists
$c \in \mathbb{R}$ such that $F(x) = G(x) + c$.

Proof. F and G are antiderivatives of $f \iff F' = G' \iff (F - G)' = 0 \iff$
$\exists\, c \in \mathbb{R}$: $F - G = c \iff \exists\, c \in \mathbb{R}$: $F(x) = G(x) + c \;\; \forall x \in I$.

Remark. This theorem holds only on an interval.

Definition. If f has an antiderivative then the set of antiderivatives of f is called
the **indefinite integral** of f:

$$\int f(x)\, dx = \{H : H'(x) = f(x)\ \forall x \in I\} = F(x) + c$$

Remark. Let $F(x) = \begin{cases} \ln x & \text{if } x > 0 \\ 4 + \ln(-x) & \text{if } x < 0 \end{cases}$ and $G(x) = \begin{cases} 3 + \ln x & \text{if } x > 0 \\ -2 + \ln(-x) & \text{if } x < 0 \end{cases}$.

$\implies F'(x) = G'(x) = \frac{1}{x} \;\; \forall x \in \mathbb{R} \setminus \{0\} = H$

$\implies F$ and G are antiderivatives of $f(x) = \frac{1}{x}$ on the set H, however,

$F(x) - G(x) = \begin{cases} -3 & \text{if } x > 0 \\ 6 & \text{if } x < 0 \end{cases}$, so their difference is not a constant.

It is important, that the above theorem holds only on an interval.

On the contrary, we use the following notation: $\int \frac{1}{x}\, dx = \ln |x| + c$

It means that $\int \frac{1}{x}\, dx = \ln x + c$ if $I \subset (0\ \infty)$ and $\int \frac{1}{x}\, dx = \ln(-x) + c$ if $I \subset (-\infty, 0)$.

Theorem. If f and g have antiderivatives on I and $c \in \mathbb{R}$ then
$f + g$, $f - g$ and $c f$ also have antiderivatives on I and

1) $\int (f \pm g) = \int f \pm \int g$

2) $\int c f = c \int f$

Basic integrals

$$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + c \;(\alpha \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^{x}\,dx = e^{x} + c$$

$$\int a^{x}\,dx = \frac{a^{x}}{\ln a} + c \;(0 < a \neq 1)$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + c$$

$$\int \frac{1}{\sin^2 x}\,dx = -\cot x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c = -\arccos x + c$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + c = -\text{arccot}\, x + c$$

$$\int \sinh x\,dx = \cosh x + c$$

$$\int \cosh x\,dx = \sinh x + c$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x + c$$

$$\int \frac{1}{\sinh^2 x}\,dx = -\coth x + c$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \text{arsinh}\, x + c$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \text{arcosh}\, x + c$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$$

Integration methods

Theorem. $\int f(a\,x+b)\,dx = \frac{F(a\,x+b)}{a} + c$, where $F' = f$ and $a \neq 0$

Proof. $\left(\frac{F(a\,x+b)}{a} + c\right)' = \frac{1}{a} F'(a\,x+b)\cdot a = f(a\,x+b)$

Example 1. $\int \sqrt{5\,x-8}\,dx = \int (5\,x-8)^{\frac{1}{2}}\,dx = \frac{(5\,x-8)^{\frac{3}{2}}}{\frac{3}{2}\cdot 5} + c$

Example 2. $\int (e^{-x} + \cos 2\,x)\,dx = \frac{e^{-x}}{-1} + \frac{\sin 2\,x}{2} + c$

Example 3. $\int \frac{5}{4\,x^2+1}\,dx = \int 5\cdot\frac{1}{(2\,x)^2+1}\,dx = 5\cdot\frac{\text{arctg}(2\,x)}{2} + c$

Example 4. $\int \frac{5}{4\,x^2+2}\,dx = \int \frac{5}{2}\cdot\frac{1}{2\,x^2+1}\,dx = \int \frac{5}{2}\cdot\frac{1}{\left(\sqrt{2}\,x\right)^2+1}\,dx = \frac{5}{2}\cdot\frac{\text{arctg}\left(\sqrt{2}\,x\right)}{\sqrt{2}} + c$

Theorem. 1) $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

2) $\int f'(x)\,(f(x))^{\alpha}\,dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq -1$

Proof. These are consequences of the differentiation rules.

Example 1. $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)+c$

Example 2. $\int \tan x\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x|+c$

Example 3. $\int \cos x \sin^3 x\,dx = \frac{\sin^4 x}{4} + c \quad (f(x)=\sin x,\ f'(x)=\cos x,\ \alpha=3)$

Example 4. $\int x\sqrt{1+x^2}\,dx = \frac{1}{2}\cdot\int 2x(1+x^2)^{\frac{1}{2}}\,dx = \frac{1}{2}\cdot\frac{(1+x^2)^{\frac{3}{2}}}{\frac{3}{2}}+c = \frac{1}{3}\cdot\sqrt{(1+x^2)^3}+c$

Integration by parts

Theorem. Assume that f and g are differentiable on the interval I and $f\cdot g'$ has an antiderivative on I. Then $f'\cdot g$ also has an antiderivative here and

$$\int f'(x)\,g(x)\,dx = f(x)\,g(x) - \int f(x)\,g'(x)\,dx$$

Proof. The right-hand side is differentiable and its derivative is

$$\left(f(x)\,g(x) - \int f(x)\,g'(x)\,dx\right)' = f'(x)\,g(x) + f(x)\,g'(x) - f(x)\,g'(x) = f'(x)\,g(x).$$

Applications

1. $g(x)$ is a polynomial of degree n and

$f'(x) = e^{ax+b},\ \sin(ax+b),\ \cos(ax+b),\ \sinh(ax+b),\ \cosh(ax+b)$

$\Longrightarrow$ the method has to be applied n times

Example 1.

$$\int x\cos 2x\,dx = \frac{\sin 2x}{2}\cdot x - \int \frac{\sin 2x}{2}\cdot 1\,dx = x\cdot\frac{\sin 2x}{2} + \frac{\cos 2x}{4} + c$$

$f'(x) = \cos 2x \ \Longrightarrow\ f(x) = \frac{\sin 2x}{2}$

$g(x) = x \ \Longrightarrow\ g'(x) = 1$

Example 2.

$$\int x^2 e^x\,dx = e^x\cdot x^2 - \int e^x\cdot 2x\,dx = e^x\cdot x^2 - \left(e^x\cdot 2x - \int e^x\cdot 2\,dx\right) =$$

$$= e^x\cdot x^2 - e^x\cdot 2x + 2e^x + c$$

$f'(x) = e^x \ \Longrightarrow\ f(x) = e^x$ $\qquad u'(x) = e^x \ \Longrightarrow\ u(x) = e^x$

$g(x) = x^2 \ \Longrightarrow\ g'(x) = 2x$ $\qquad v(x) = 2x \ \Longrightarrow\ v'(x) = 2$

2. $f'(x)$ is a polynomial and

$g(x) = \ln x,\ \arcsin x,\ \arccos x,\ \arctan x,\ \text{arccot}\,x,\ \text{arsinh}\,x,\ \ldots$

Example 1.

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int 1 \, dx = x \ln x - x + c$$

$$f'(x) = 1 \implies f(x) = x$$

$$g(x) = \ln x \implies g'(x) = \frac{1}{x}$$

Example 2.

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

$$f'(x) = x \implies f(x) = \frac{x^2}{2}$$

$$g(x) = \ln x \implies g'(x) = \frac{1}{x}$$

Example 3.

$$\int \operatorname{arctg} x \, dx = \int 1 \cdot \operatorname{arctg} x \, dx = x \cdot \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2} \, dx$$

$$= x \cdot \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx = x \cdot \operatorname{arctg} x - \frac{1}{2} \ln\left(1+x^2\right) + c$$

$$f'(x) = 1 \implies f(x) = x$$

$$g(x) = \operatorname{arctg} x \implies g'(x) = \frac{1}{1+x^2}$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + c$$

Example 4.

$$\int \arcsin x \, dx = \int 1 \cdot \arcsin x \, dx = x \cdot \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= x \cdot \arcsin x - \int x \cdot \left(1-x^2\right)^{-\frac{1}{2}} dx = x \cdot \arcsin x + \frac{1}{2} \cdot \int (-2x)\left(1-x^2\right)^{-\frac{1}{2}} dx =$$

$$= x \cdot \arcsin x + \frac{1}{2} \frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = x \arcsin x + \sqrt{1-x^2} + c$$

$$f'(x) = 1 \implies f(x) = x$$

$$g(x) = \arcsin x \implies g'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\int f'(x) \cdot (f(x))^{\alpha} \, dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq 0$$

3. $f'(x)$ and $g(x)$ are both one of the following functions:

e^{ax+b}, $\sin(ax+b)$, $\cos(ax+b)$, $\sinh(ax+b)$, $\cosh(ax+b)$

$\implies$ the method has to be applied twice

Example. $I=\int e^x \sin x\, dx = ?$

$I=\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx$

$f'(x)=e^x \implies f(x)=e^x$
$g(x)=\sin x \implies g'(x)=\cos x$

$I=\int e^x \sin x\, dx = e^x(-\cos x) - \int e^x(-\cos x)\, dx$

$f'(x)=\sin x \implies f(x)=-\cos x$
$g(x)=e^x \implies g'(x)=e^x$

(1) $I=e^x \sin x - \int e^x \cos x\, dx$
(2) $I=e^x(-\cos x) + \int e^x \cos x\, dx$

(1) + (2) $\implies 2I = e^x \sin x - e^x \cos x \implies I=\frac{1}{2} e^x(\sin x - \cos x)+c$

Powers of sin x and cos x

Odd powers of sin x and cos x:
$\sin^{2n+1} x = \sin x \cdot (\sin^2 x)^n = \sin x \cdot (1-\cos^2 x)^n = \ldots$
$\cos^{2n+1} x = \cos x \cdot (\cos^2 x)^n = \cos x \cdot (1-\sin^2 x)^n = \ldots$

Even powers of sin x and cos x:
$\sin^{2n} x = (\sin^2 x)^n = \left(\frac{1-\cos 2x}{2}\right)^n + \ldots$
$\cos^{2n} x = (\cos^2 x)^n = \left(\frac{1+\cos 2x}{2}\right)^n + \ldots$

Example 1.
$I=\int \sin^3 x\, dx = \int \sin x \sin^2 x\, dx = \int \sin x\,(1-\cos^2 x)\, dx =$
$= \int (\sin x - \sin x \cos^2 x)\, dx = -\cos x + \frac{\cos^3 x}{3} + c$
$\int f'(x)\cdot(f(x))^\alpha\, dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c,\ \ \alpha \neq 0$

Example 2.
$I=\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1-\sin^2 x)^2 \cos x\, dx =$
$= \int (1-2\sin^2 x + \sin^4 x)\cos x\, dx = \int (\cos x - 2\sin^2 x \cos x + \sin^4 x \cos x)\, dx$
$= \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + c$
$\int f'(x)\cdot(f(x))^\alpha\, dx = \frac{(f(x))^{\alpha+1}}{\alpha+1} + c,\ \ \alpha \neq 0$

Example 3.
$I=\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$

Example 4.

$$I=\int\cos^4 x\,dx=\int(\cos^2 x)^2\,dx=\int\left(\frac{1+\cos 2x}{2}\right)^2 dx=$$

$$=\int\frac{1}{4}\cdot(1+2\cos 2x+\cos^2 2x)\,dx=$$

$$\cos^2 2x=\frac{1+\cos 4x}{2}$$

$$=\int\left(\frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{8}+\frac{1}{8}\cos 4x\right)dx=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c$$

Products of powers of sin *x* and cos *x*:

$\int\sin^n x\cos^m x\,dx=?\quad n, m\in\mathbb{N}^+$

1. If *n* or *m* is odd:

$$\int\sin^3 x\cos^4 x\,dx=\int\sin x\sin^2 x\cos^4 x\,dx=\int\sin x\,(1-\cos^2 x)\cos^4 x\,dx=$$

$$=\int\sin x\,(\cos^4 x-\cos^6 x)\,dx=-\frac{\cos^5 x}{5}+\frac{\cos^7 x}{7}+c$$

2. If *n* and *m* are even:

$$\int\sin^2 x\cos^4 x\,dx=\int(1-\cos^2 x)\cos^4 x\,dx=\int(\cos^4 x-\cos^6 x)\,dx=\ldots$$

Integrals of rational functions

Polynomial division

Definition: If $p_1(x)$ and $p_2(x)$ are polynomials then the function $R(x)=\frac{p_1(x)}{p_2(x)}$

is called a rational function.

Statement. Polynomials can be divided in the following sense:

$p_1(x)=q(x)\,p_2(x)+r(x)$

where $\deg r<\deg p_2$. If $\deg p_1<\deg p_2$ then $q=0$ and $r=p_1$.

The polynomial q is the quotient and r is the remainder.

Example 1. Divide $p_1(x)=x^3-4x^2+5x-8$ by $p_2(x)=x^2+3$

$(x^3-4x^2+5x-8)\;:(x^2+3)=x-4$ ($x^3 : x^2 = x$, then multiply $p_2(x)$ by x^2)

$-(x^3\quad +3x)$

$0\,-4x^2+2x-8$ ($-4x^2 : x^2=-4$, then multiply $p_2(x)$ by -4)

$-(-4x^2\quad -12)$

$0\,+\,2x\,+4$ $\Longrightarrow$ the remainder is $2x+4$, since $\deg(2x+4)<\deg(x^2+3)$

$$\Longrightarrow x^3-4x^2+5x-8=(x^2+3)(x-4)+(2x+4)$$

$$\Longrightarrow \frac{x^3-4x^2+5x-8}{x^2+3}=(x-4)+\frac{2x+4}{x^2+3}$$

Example 2. Divide $p_1(x) = x^4$ by $p_2(x) = x^2 + x - 2$

$x^4 \quad : (x^2 + x - 2) = x^2 - x + 3$ $\quad (x^4 : x^2 = x^2$, then multiply $p_2(x)$ by x^2)

$-(x^4 + x^3 - 2x^2)$ (the product is subtracted from the line above)

$0 - x^3 + 2x^2$ $\quad (-x^3 : x^2 = -x$, then multiply $p_2(x)$ by $-x$)

$-(-x^3 - x^2 + 2x)$

$0 + 3x^2 - 2x$ $\quad (3x^2 : x^2 = 3$, then multiply $p_2(x)$ by 3)

$-(3x^2 + 3x - 6)$

$0 - 5x + 6$ $\quad \Longrightarrow$ the remainder is $-5x + 6$, since $\deg(-5x + 6) < \deg(x^2 - x + 3)$

$$\Longrightarrow x^4 = (x^2 + x - 2)(x^2 - x + 3) + (-5x + 6)$$

$$\Longrightarrow \frac{x^4}{x^2 + x - 2} = (x^2 - x + 3) + \frac{-5x + 6}{x^2 + x - 2}$$

Integration of rational functions

1st step. If $R(x) = \dfrac{T(x)}{Q(x)}$ and $\deg T(x) \geq \deg Q(x)$ then with polynomial division we bring it to the form $R(x) = E(x) + \dfrac{P(x)}{Q(x)}$, where $E(x)$ is a polynomial and $\deg P(x) < \deg Q(x)$.

2nd step. The denominator can be written as

$Q(x) = (x - a_1)^{\alpha_1} \ldots (x - a_r)^{\alpha_r} (x^2 + b_1 x + c_1)^{\beta_1} \ldots (x^2 + b_s x + c_s)^{\beta_s}$

where $b_i^2 - 4c_i < 0$ and $\alpha_1, \ldots, \alpha_r$ are the multiplicities of the real roots and $\beta_1, \ldots, \beta_s$ are the multiplicities of the complex roots.

3rd step. Partial fraction decomposition. It means that to each term in the above form of $Q(x)$ we assign an elementary fraction (or partial fraction) such that the sum of these fractions is equal to $\dfrac{P(x)}{Q(x)}$. This decomposition is unique.

	Factor in the denominator	Term in the partial fraction decomposition
Single real root :	$x - a$	$\dfrac{A}{x - a}$
Multiple real root :	$(x - a)^k$	$\dfrac{A_1}{x - a} + \dfrac{A_2}{(x - a)^2} + \ldots + \dfrac{A_k}{(x - a)^k}$
Single complex roots $(b^2 - 4c < 0)$:	$x^2 + bx + c$	$\dfrac{Bx + C}{x^2 + bx + c}$
Multiple complex roots $(b^2 - 4c < 0)$:	$(x^2 + bx + c)^k$	$\dfrac{B_1 x + C_1}{x^2 + bx + c} + \dfrac{B_2 x + C_2}{(x^2 + bx + c)^2} + \ldots + \dfrac{B_k x + C_k}{(x^2 + bx + c)^k}$

4th step. We integrate the polynomial $E(x)$ and the partial fractions term by term.

Integration of the elementary fractions

(1) $\int \frac{A}{x-1}\,dx = A\ln|x-a| + c$

(2) $k\in\mathbb{N},\ k>1$: $\int \frac{A}{(x-1)^k}\,dx = \frac{-kA}{(x-1)^{k-1}} + c$

(3) $b^2-4c<0$: $\int \frac{Bx+C}{x^2+bx+c}\,dx = \int \frac{B}{2}\frac{2x+b}{x^2+bx+c}\,dx + \int \frac{-\frac{Bb}{2}+C}{x^2+bx+c}\,dx$

(i) $\int \frac{2x+b}{x^2+bx+c}\,dx = \ln(x^2+bx+c) + \text{constant}$

(ii) $\int \frac{1}{x^2+bx+c}\,dx = \int \frac{1}{\left(x+\frac{b}{2}\right)^2+d^2}\,dx = \frac{1}{d^2}\int \frac{1}{\left(\frac{x+\frac{b}{2}}{d}\right)^2+1}\,dx = \frac{1}{d^2}\,\frac{\arctan\left(\frac{x+\frac{b}{2}}{d}\right)}{\frac{1}{d}} + \text{constant}$

where $d=\sqrt{c-\frac{b^2}{4}}$.

(4) $b^2-4c<0,\ k\in\mathbb{N},\ k>1$: $\int \frac{Bx+C}{(x^2+bx+c)^k}\,dx = \int \frac{B}{2}\frac{2x+b}{(x^2+bx+c)^k}\,dx + \int \frac{-\frac{Bb}{2}+C}{(x^2+bx+c)^k}\,dx$

(i) $\int \frac{2x+b}{(x^2+bx+c)^k}\,dx = \frac{1}{1-k}\,\frac{1}{(x^2+bx+c)^{k-1}} + \text{constant}$

(ii)* $\int \frac{1}{(x^2+bx+c)^k}\,dx = \int \frac{1}{\left(\left(x+\frac{b}{2}\right)^2+d^2\right)^k}\,dx$, where $d=\sqrt{c-\frac{b^2}{4}}$

If $\int \frac{1}{(x^2+1)^n}\,dx = F_n(x)+c$ then $\int \frac{1}{\left(\left(x+\frac{b}{2}\right)^2+d^2\right)^k}\,dx = \frac{1}{d^{2n-1}}F_n\left(\frac{x}{d}+\frac{b}{2d}\right)+c.$

Remark. If $\int \frac{1}{(x^2+1)^n}\,dx = F_n(x)+c$ then for $n\ge 1$: $F_{n+1}(x) = \frac{1}{2n}\frac{x}{(x^2+1)^n} + \frac{2n-1}{2n}F_n(x)+c.$

For example, $F_1(x) = \arctan x + c$

$$F_2(x) = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\arctan x + c$$

$$F_3(x) = \frac{1}{4}\frac{x}{x^2+1} + \frac{3}{8}\frac{x}{x^2+1} + \frac{3}{8}\arctan x + c$$

Examples

Example 1. $I = \int \frac{x+1}{x^2+3x}\,dx = ?$

Solution. The denominator has two distinct real roots: $x_1=0$, $x_2=-3$.

Partial fraction decomposition:

$$\frac{x+1}{x^2+3x} = \frac{x+1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3} = \frac{A(x+3)+Bx}{x(x+3)}$$

$\Longrightarrow\ x+1=A(x+3)+B\,x$

We want to find A and B such that this equation holds **for all** $x\in\mathbb{R}$.

1st method (comparison of the coefficients)

$x+1=(A+B)\,x+3\,A$ holds **for all** $x\in\mathbb{R}$ if and only if

$$\begin{array}{l} A+B=1 \\ 3\,A=1 \end{array}\ \Longrightarrow\ A=\frac{1}{3},\ B=\frac{2}{3}$$

2nd method (substitution) (it is worth substituting the roots of the denominator)

$x+1=A(x+3)+B\,x$

if $x=0\ \Longrightarrow\quad 1=A\cdot 3+B\cdot 0\ \Longrightarrow\ A=\frac{1}{3}$

if $x=-3\ \Longrightarrow\ -2=A\cdot 0+B\cdot(-3)\ \Longrightarrow\ B=\frac{2}{3}$

$$\Longrightarrow\ I=\int\frac{x+1}{x^2+3\,x}\,dx=\int\left(\frac{1}{3}\cdot\frac{1}{x}+\frac{2}{3}\cdot\frac{1}{x+3}\right)dx=\frac{1}{3}\ln|x|+\frac{2}{3}\ln|x+3|+c$$

Example 2. $I=\int\frac{x+5}{x^2+6\,x+9}\,dx=?$

Solution. The denominator has multiple real roots: $x_{1,2}=-3$.

Partial fraction decomposition:

$$\frac{x+5}{x^2+6\,x+9}=\frac{x+5}{(x+3)^2}=\frac{A}{x+3}+\frac{B}{(x+3)^2}=\frac{A(x+3)+B}{(x+3)^2}$$

$\Longrightarrow\ x+5=A(x+3)+B$

1st method: $x+5=A\,x+(3\,A+B)\ \Longrightarrow\ \begin{array}{l} A=1 \\ 3\,A+B=5 \end{array}\ \Longrightarrow\ A=1,\ B=2$

2nd method: $x+5=A(x+3)+B$

$x=-3\ \Longrightarrow\quad 2=0+B\ \Longrightarrow B=2$

$x=0\quad\Longrightarrow\quad 5=3\,A+B\ \Longrightarrow\ A=1$

$$\Longrightarrow\ I=\int\frac{x+5}{x^2+6\,x+9}\,dx=\int\left(\frac{1}{x+3}+\frac{2}{(x+3)^2}\right)dx=\int\left(\frac{1}{x+3}+2\,(x+3)^{-2}\right)dx=$$

$$=\ln|x+3|+2\,\frac{(x+3)^{-1}}{-1}+c=\ln|x+3|-\frac{2}{x+3}+c$$

Example 3. $I = \int \frac{1}{(x-1)^2\,(x^2+1)}\,dx = ?$

Solution: The roots of the denominator are:

$x_{1,2} = 1$ (multiple real roots), $x_{3,4} = \pm i$ (simple complex roots)

Partial fraction decomposition:

$$\frac{1}{(x-1)^2\,(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C\,x+D}{x^2+1} = \frac{A(x-1)\,(x^2+1) + B(x^2+1) + (C\,x+D)\,(x-1)^2}{(x-1)^2\,(x^2+1)}$$

$$\Longrightarrow\ 1 = A(x-1)\,(x^2+1) + B(x^2+1) + (C\,x+D)\,(x-1)^2$$

Substitutions:

$x = 1 \Longrightarrow\ 1 = 0 + B + 0 \Longrightarrow\ B = 1$

$x = 0 \Longrightarrow\ 1 = -A + B + D$

$x = -1 \Longrightarrow\ 1 = -4\,A + 2\,B - 4\,C + 4\,D$

$x = 2 \Longrightarrow\ 1 = 5\,A + 5\,B + 2\,C + D$

Homework: the solution of this equation system is

$C = \frac{1}{2},\ D = 0,\ A = -\frac{1}{2},\ B = \frac{1}{2}$

$$\Longrightarrow\ I = \int \frac{1}{(x-1)^2\,(x^2+1)}\,dx = \int\left(-\frac{1}{2}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{(x-1)^2} + \frac{1}{2}\cdot\frac{x}{x^2+1}\right)dx =$$

$$= \int\left(-\frac{1}{2}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot(x-1)^{-2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2\,x}{x^2+1}\right)dx = \left(\text{the last term has the form } \frac{f'}{f}\right)$$

$$= -\frac{1}{2}\ln\left|x-1\right| + \frac{1}{2}\cdot\frac{(x-1)^{-1}}{-1} + \frac{1}{4}\ln(x^2+1) + c$$

Integration by change of variables

The substitution formula

Theorem. Assume that g is differentiable on the interval I, f is defined on $J = g(I)$ and f has a primitive function on J. Then $(f \circ g)\,g'$ also has a primitive function on I and

$$\int f(g(x))\cdot g'(x)\,dx = F(g(x)) + c \quad \text{where} \quad \int f(x)\,dx = F(x) + c$$

Proof. $(F(g(x)) + c)' = F'(g(x))\cdot g'(x) = f(g(x))\cdot g'(x)$

Remark. If g is invertible then the above formula can be written in the form

$$\int f(g(t))\cdot g'(t)\,dt\ \big|_{t=g^{-1}(x)} = \int f(x)\,dx$$

Examples

Example 1. $I = \int \sin x\, e^{\cos x}\, dx = ?$

Substitution: $t = \cos x \implies x = x(t) = \arccos t$

$$x'(t) = \frac{dx}{dt} = -\frac{1}{\sqrt{1-t^2}} \implies dx = -\frac{1}{\sqrt{1-t^2}}\, dt$$

$$\sin x = \sqrt{\sin^2 x} = \sqrt{1-\cos^2 x} = \sqrt{1-t^2}$$

$$I = \int \sin x\, e^{\cos x}\, dx = \int \sqrt{1-t^2}\, e^t \cdot \left(-\frac{1}{\sqrt{1-t^2}}\right) dt = \int -e^t\, dt = -e^t + c = -e^{\cos x} + c$$

Remark. $\int e^{f(x)} \cdot f'(x)\, dx = e^{f(x)} + c$

Example 2. $I = \int x^2 \sin x^3\, dx = ?$

Substitution: $t = x^3 \implies x = x(t) = \sqrt[3]{t} = t^{\frac{1}{3}} \implies x^2 = t^{\frac{2}{3}}$

$$x'(t) = \frac{dx}{dt} = \frac{1}{3}\, t^{-\frac{2}{3}} \implies dx = \frac{1}{3}\, t^{-\frac{2}{3}}\, dt$$

$$I = \int x^2 \sin\left(x^3\right) dx = \int t^{\frac{2}{3}} \sin t \cdot \frac{1}{3}\, t^{-\frac{2}{3}}\, dt = \int \frac{1}{3} \sin t\, dt = -\frac{1}{3} \cos t + c = -\frac{1}{3} \cos x^3 + c$$

Example 3. $I = \int \sin^4 x \cos dx = ?$

1st solution. $\int f'(x)\, f^\alpha(x)\, dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + c \quad (\alpha \neq -1)$

Here: $f(x) = \sin x,\ f'(x) = \cos x,\ \alpha = 4 \implies$

$$I = \int \sin^4 x \cos x\, dx = \frac{\sin^5 x}{5} + c$$

2nd solution.

Substitution: $t = \sin x \implies x = x(t) = \arcsin t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}} \implies dx = \frac{1}{\sqrt{1-t^2}}\, dt$$

$$\cos x = \sqrt{\cos^2 x} = \sqrt{1-\sin^2 x} = \sqrt{1-t^2}$$

$$I = \int \sin^4 x \cos x\, dx = \int t^4 \sqrt{1-t^2} \cdot \frac{1}{\sqrt{1-t^2}}\, dt = \int t^4\, dt = \frac{t^5}{5} + c = \frac{\sin^5 x}{5} + c$$

Rational functions of e^x

Statement. The integral $\int R(e^x)\,\mathrm{dx}$ (where R is a rational function) can be transformed to the integral of a rational function $t = e^x$.

Example 1. $I = \int \frac{e^{2x}}{e^x + 1}\,\mathrm{dx} = ?$

Substitution: $t = e^x \implies x = x(t) = \ln t$

$$x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{1}{t} \implies \mathrm{dx} = \frac{1}{t}\,\mathrm{dt}$$

$$I = \int \frac{e^{2x}}{e^x + 1}\,\mathrm{dx} = \int \frac{t^2}{t+1} \cdot \frac{1}{t}\,\mathrm{dt} = \int \frac{t}{t+1}\,\mathrm{dt} = \int \frac{(t+1)-1}{t+1}\,\mathrm{dt} = \int \left(1 - \frac{1}{t+1}\right)\mathrm{dt} =$$
$$= t - \ln|t+1| + c = e^x - \ln(e^x + 1) + c$$

Example 2. $I = \int \frac{4}{e^{2x} - 4}\,\mathrm{dx} = ?$

Substitution: $t = e^x \implies x = x(t) = \ln t$

$$x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{1}{t} \implies \mathrm{dx} = \frac{1}{t}\,\mathrm{dt}$$

$$I = \int \frac{4}{e^{2x} - 4}\,\mathrm{dx} = \int \frac{4}{t^2 - 4} \cdot \frac{1}{t}\,\mathrm{dt} = \int \frac{4}{t(t-2)(t+2)}\,\mathrm{dt}$$

Partial fraction decomposition:

$$\frac{4}{t(t-2)(t+2)} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-2}$$

$\implies$ $4 = A(t+2)(t-2) + B\,t(t-2) + C\,t(t+2)$

$t = 0$: $4 = -4A + 0 + 0 \implies A = -1$

$t = -2$: $4 = 0 + 8B + 0 \implies B = \frac{1}{2}$

$t = 2$: $4 = 0 + 0 + 8C \implies C = \frac{1}{2}$

$$\implies I = \int \left(-\frac{1}{t} + \frac{1}{2}\frac{1}{t+2} + \frac{1}{2}\frac{1}{t-2}\right)\mathrm{dt} = -\ln|t| + \frac{1}{2}\ln|t+2| + \frac{1}{2}\ln|t-2| + c =$$
$$= -\ln e^x + \frac{1}{2}\ln(e^x + 2) + \frac{1}{2}\ln|e^x - 2| + c$$

Some integrals with roots

Remark. In the following cases $R(u, v)$ denotes a two-variable rational function, that is, $R(u, v) = \frac{P(u, v)}{Q(u, v)}$, $P(u, v) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\,u^i\,v^j$, $Q(u, v) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\,u^i\,v^j$, $n \in \mathbb{N}$.

1. The integral $\int R\left(x, \sqrt[n]{\frac{a\,x+b}{c\,x+d}}\right) dx$ (where $c^2+d^2 \neq 0,\ a\,d \neq b\,c$) can be transformed to the integral of a rational function with the substitution $t = \sqrt[n]{\frac{a\,x+b}{c\,x+d}}$.

Example 1. $I = \int \frac{1}{x^2} \sqrt[3]{\frac{x+1}{x}}\, dx = ?$

Substitution: $t = \sqrt[3]{\frac{x+1}{x}} \Longrightarrow x+1 = t^3\,x$

$$x = x(t) = \frac{1}{t^3-1}$$

$$x'(t) = \frac{dx}{dt} = -\frac{3\,t^2}{(t^3-1)^2} \Longrightarrow dx = -\frac{3\,t^2}{(t^3-1)^2}\,dt$$

$$I = \int \frac{1}{x^2} \sqrt[3]{\frac{x+1}{x}}\, dx = \int (t^3-1)^2 \cdot t \cdot \frac{-3\,t^2}{(t^3-1)^2}\,dt = \int -3\,t^3\,dt = -\frac{3}{4}\,t^4 + c = -\frac{3}{4}\left(\frac{x+1}{x}\right)^{\frac{4}{3}} + c$$

Example 2. $I = \int \sqrt{\frac{x-3}{x-1}}\, dx = ?$

Substitution: $t = \sqrt{\frac{x-3}{x-1}} \Longrightarrow t^2(x-1) = x-3 \Longrightarrow x\left(t^2-1\right) = t^2-3$

$$x = x(t) = \frac{t^2-3}{t^2-1} = \frac{t^2-1-2}{t^2-1} = 1 - \frac{2}{t^2-1}$$

$$x'(t) = \frac{dx}{dt} = \frac{4\,t}{(t^2-1)^2} \Longrightarrow dx = \frac{4\,t}{(t^2-1)^2}\,dt$$

$$I = \int \sqrt{\frac{x-3}{x-1}}\, dx = \int t \cdot \frac{4\,t}{(t^2-1)^2}\,dt = \int \frac{4\,t^2}{(t-1)^2\,(t+1)^2}\,dt$$

Partial fraction decomposition: $\frac{4\,t^2}{(t-1)^2\,(t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$

$\Longrightarrow A = 1,\ B = 1,\ C = -1,\ D = 1$

$$\Longrightarrow I = \int \left(\frac{1}{t-1} + \frac{1}{(t-1)^2} - \frac{1}{t+1} + \frac{1}{(t+1)^2}\right) dt = \ln\left|\,t-1\,\right| - \frac{1}{t-1} - \ln\left|\,t+1\,\right| - \frac{1}{t+1} + c$$

$$= -\frac{2\,t}{t^2-1} + \ln\left|\,\frac{t-1}{t+1}\,\right| + c = -\frac{2\sqrt{\frac{x-3}{x-1}}}{\frac{x-3}{x-1}-1} + \ln\left|\,\frac{\sqrt{\frac{x-3}{x-1}}-1}{\sqrt{\frac{x-3}{x-1}}+1}\,\right| + c$$

2. The integral $\int R\left(x, \sqrt[n]{a\,x+b}\right) dx$ can be transformed to the integral of a rational function with the substitution $t = \sqrt[n]{a\,x+b}$.

Example 1. $I = \int x\sqrt{5x+3}\,\mathrm{dx} = ?$

Substitution: $t = \sqrt{5x+3} \implies x = x(t) = \dfrac{t^2-3}{5}$

$$x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{2}{5}t \implies \mathrm{dx} = \frac{2}{5}t\,\mathrm{dt}$$

$$I = \int x\sqrt{5x+3}\,\mathrm{dx} = \int \frac{t^2-3}{5}\cdot t\cdot\frac{2}{5}t\,\mathrm{dt} = \int \frac{2}{25}(t^4-3t^2)\,\mathrm{dt} = \frac{2}{25}\left(\frac{t^5}{5}-t^3\right)+c =$$

$$= \frac{2}{25}\left(\frac{\sqrt{(5x+3)^5}}{5} - \sqrt{(5x+3)^3}\right)+c$$

Example 2. $I = \int \dfrac{1}{\sqrt{x}+1}\,\mathrm{dx} = ?$

Substitution: $t = \sqrt{x} \implies x = x(t) = t^2$

$$x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = 2t \implies \mathrm{dx} = 2t\,\mathrm{dt}$$

$$I = \int \frac{1}{\sqrt{x}+1}\,\mathrm{dx} = \int \frac{2t}{t+1}\,\mathrm{dt} = \int \frac{2(t+1)-1}{t+1}\,\mathrm{dt} = \int 2\cdot\left(1-\frac{1}{t+1}\right)\mathrm{dt} =$$

$$= 2t - 2\ln|1+t| + c = 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right)+c$$

3. In the integral $\int R\left(x, \sqrt{a x^2 + b x + c}\right)\mathrm{dx}$ $(a \neq 0)$ after completing the square under the root sign, either of the following substitutions can be used:

$\sqrt{1-A^2}$ $\quad A = \sin t,\ \ t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (or $A = \cos t,\ \ t \in [0, \pi]$)

$\sqrt{B^2+1}$ $\quad B = \sinh t$

$\sqrt{C^2-1}$ $\quad C = \cosh t$

Identities for taking the square root: $\cos^2 t + \sin^2 t = 1$, $\cosh^2 t - \sinh^2 t = 1$.

Example 1. $I = \int \sqrt{4-x^2}\,\mathrm{dx} = ?$

Substitution: $x = x(t) = 2\sin t \implies t = \arcsin\left(\dfrac{x}{2}\right)$

$$x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = 2\cos t \implies \mathrm{dx} = 2\cos t\,\mathrm{dt}$$

$$I = \int \sqrt{4-x^2}\,\mathrm{dx} = \int \sqrt{4-4\sin^2 t}\cdot 2\cos t\,\mathrm{dt} = \int 2\sqrt{1-\sin^2 t}\cdot 2\cos t\,\mathrm{dt} = \int 4\sqrt{\cos^2 t}\cdot\cos t\,\mathrm{dt}$$

$$= \int 4\cos^2 t\,\mathrm{dt} = \int 4\cdot\frac{1+\cos 2t}{2}\,\mathrm{dt} = \int 2\cdot(1+\cos 2t)\,\mathrm{dt} = 2\left(t+\frac{\sin 2t}{2}\right)+c =$$

$$= 2t + \sin 2t + c = 2\cdot\arcsin\left(\frac{x}{2}\right) + 2\cdot\frac{x}{2}\sqrt{1-\left(\frac{x}{2}\right)^2} + c$$

Identities:

$\cos^2 x + \sin^2 x = 1 \qquad \Longrightarrow \cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos^2 x - \sin^2 x = \cos 2x$

$\sin 2t = 2 \sin t \cos t = 2 \sin t \sqrt{\cos^2 t} = 2 \sin t \sqrt{1 - \sin^2 t} = 2 \cdot \frac{x}{2} \cdot \sqrt{1 - \left(\frac{x}{2}\right)^2}$

Example 2. $I = \int \frac{x^2}{\sqrt{9 - x^2}} \, \mathrm{dx} = ?$

Substitution: $x = 3 \sin t \implies t = \arcsin\left(\frac{x}{3}\right)$

$$x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = 3 \cos t \implies \mathrm{dx} = 3 \cos t \, \mathrm{dt}$$

$$I = \int \frac{x^2}{\sqrt{9 - x^2}} \, \mathrm{dx} = \int \frac{(3 \sin t)^2}{\sqrt{9 - (3 \sin t)^2}} \cdot 3 \cos t \, \mathrm{dt} = \int \frac{9 \sin^2 t}{\sqrt{9 \cdot (1 - \sin^2 t)}} \cdot 3 \cos t \, \mathrm{dt} =$$

$$= \int \frac{9 \sin^2 t}{\sqrt{\cos^2 t}} \cos t \, \mathrm{dt} = \int \frac{9 \sin^2 t}{\cos t} \cos t \, \mathrm{dt} = \int 9 \sin^2 t \, \mathrm{dt} = \int 9 \cdot \frac{1 - \cos 2t}{2} \, \mathrm{dt} =$$

$$= \frac{9}{2} \cdot \left(t - \frac{\sin 2t}{2}\right) + c = \frac{9}{2} t - \frac{9}{4} \sin 2t + c = \frac{9}{2} \arcsin\left(\frac{x}{3}\right) - \frac{9}{4} \cdot 2 \cdot \frac{x}{3} \sqrt{1 - \left(\frac{x}{3}\right)^2} + c =$$

Identities:

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$\sin 2t = 2 \sin t \cos t = 2 \sin t \sqrt{\cos^2 t} = 2 \sin t \sqrt{1 - \sin^2 t} = 2 \cdot \frac{x}{3} \cdot \sqrt{1 - \left(\frac{x}{3}\right)^2}$

Rational functions of $\sin x$ and $\cos x$

Statement. The integral $\int R(\sin x, \cos x) \, \mathrm{dx}$ where R is a rational function can be transformed to the integral of a rational function with the substitution

$t = \tan \frac{x}{2}$

$\implies x = 2 \arctan t, \; \mathrm{dx} = \frac{2}{1 + t^2} \mathrm{dt}, \; \sin x = \frac{2t}{1 + t^2}, \; \cos x = \frac{1 - t^2}{1 + t^2}$

Remark. $t = \tan \frac{x}{2} \implies x = x(t) = 2 \arctan t \implies \frac{\mathrm{dx}}{\mathrm{dt}} = x'(t) = \frac{2}{1 + t^2}$

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1 \implies \cos^2 x = \frac{1}{1 + \tan^2 x}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \tan \frac{x}{2} \cos^2 \frac{x}{2} = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2t}{1 + t^2}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos^2 \frac{x}{2} \left(1 - \tan^2 \frac{x}{2}\right) = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1 - t^2}{1 + t^2}$$

Example 1. $I = \int \frac{1}{\sin x}\,\mathrm{dx} = ?$

Substitution: $t = \tan\frac{x}{2} \implies \sin x = \frac{2t}{1+t^2},\ \mathrm{dx} = \frac{2}{1+t^2}\,\mathrm{dt}$

$$I = \int \frac{1}{\sin x}\,\mathrm{dx} = \int \frac{1+t^2}{2t}\,\frac{2}{1+t^2}\,\mathrm{dt} = \int \frac{1}{t}\,\mathrm{dt} = \ln|t| + c = \ln\left|\tan\frac{x}{2}\right| + c$$

Example 2. $I = \int \frac{1}{\cos x}\,\mathrm{dx} = ?$

Substitution: $t = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right)$

$$I = \int \frac{1}{\cos x}\,\mathrm{dx} = \int \frac{1}{\sin\left(\frac{\pi}{2} - x\right)}\,\mathrm{dx} = -\int \frac{1}{\sin\left(x - \frac{\pi}{2}\right)}\,\mathrm{dx} = -\int \frac{1}{t}\,\mathrm{dt} = -\ln\left|\tan\left(\frac{x}{2} - \frac{\pi}{4}\right)\right| + c$$

Example 3. $I = \int \frac{1}{1+\cos x}\,\mathrm{dx} = ?$

Substitution: $t = \tan\frac{x}{2} \implies \cos x = \frac{1-t^2}{1+t^2},\ \mathrm{dx} = \frac{2}{1+t^2}\,\mathrm{dt}$

$$I = \int \frac{1}{1+\cos x}\,\mathrm{dx} = \int \frac{1}{1+\frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,\mathrm{dt} = \int 1\,\mathrm{dt} = t + c = \tan\frac{x}{2} + c$$

Example 4. $I = \int \frac{1}{\sin x(1+\cos x)}\,\mathrm{dx} = ?$

Substitution: $t = \tan\frac{x}{2} \implies \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ \mathrm{dx} = \frac{2}{1+t^2}\,\mathrm{dt}$

$$I = \int \frac{1}{\sin x(1+\cos x)}\,\mathrm{dx} = \int \frac{1}{\frac{2t}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)}\,\frac{2}{1+t^2}\,\mathrm{dt} = \int \frac{1+t^2}{2t}\,\mathrm{dt}$$

$$= \int\left(\frac{1}{2t} + \frac{t}{2}\right)\mathrm{dt} = \frac{1}{2}\ln|t| + \frac{t^2}{4} + c = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + \frac{1}{4}\tan^2\frac{x}{2} + c$$

Example 5*. $I = \int \frac{1+\sin x}{1-\cos x}\,\mathrm{dx} = ?$

Substitution: $t = \tan\frac{x}{2} \implies \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ \mathrm{dx} = \frac{2}{1+t^2}\,\mathrm{dt}$

$$I = \int \frac{1+\sin x}{1-\cos x}\,\mathrm{dx} = \int \frac{1+\frac{2t}{1+t^2}}{1-\frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,\mathrm{dt} = \int \frac{t^2+2t+1}{t^2(t^2+1)}\,\mathrm{dt} =$$

Partial fraction decomposition:

$$\frac{t^2+2t+1}{t^2(t^2+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+1} \implies A = 2,\ B = 1,\ C = -2,\ D = 0$$

$$I=\int\left(\frac{2}{t}+\frac{1}{t^2}-\frac{2t}{t^2+1}\right)dt=2\ln\left|\tan\frac{x}{2}\right|-\frac{1}{\tan\frac{x}{2}}-\ln\left(1+\tan^2\frac{x}{2}\right)+c$$

Remark. For even powers of $\sin x$ and $\cos x$ the following transformations are better:

$$1+\cot^2 x=\frac{\sin^2 x+\cos^2 x}{\sin^2 x}=\frac{1}{\sin^2 x}\implies \sin^2 x=\frac{1}{1+\cot^2 x}$$
$$1+\tan^2 x=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}\implies \cos^2 x=\frac{1}{1+\tan^2 x}$$

Example 1*. $I=\int\frac{1}{\sin^6 x}\,dx=?$

$$I=\int\frac{1}{\sin^6 x}\,dx=\int\left(\frac{1}{\sin^2 x}\right)^2\frac{1}{\sin^2 x}\,dx=\int(1+\cot^2 x)^2\frac{1}{\sin^2 x}\,dx$$

Substitution: $y=\cot x\implies \frac{dy}{dx}=-\frac{1}{\sin^2 x}$

$$I=-\int(1+y^2)^2\,dy=-\int(1+2y^2+y^4)\,dy=-\left(y+\frac{2}{3}y^3+\frac{y^5}{5}\right)+c$$
$$=-\left(\cot x+\frac{2}{3}(\cot x)^3+\frac{(\cot x)^5}{5}\right)+c$$

Example 2*. $I=\int\frac{1}{2+\sin^2 x}\,dx=?$

$$I=\int\frac{1}{2+\sin^2 x}\,dx=\int\frac{1}{2+\frac{1}{1+\cot^2 x}}\,dx=\int\frac{1+\cot^2 x}{3+2\cot^2 x}\,dx$$

Substitution: $y=\cot x$

$$x=\operatorname{arccot} y\implies dx=-\frac{1}{1+y^2}\,dy$$

$$I=-\int\frac{1+y^2}{3+2y^2}\,\frac{1}{1+y^2}\,dy=-\int\frac{1}{3+2y^2}\,dy=-\frac{1}{3}\int\frac{1}{1+\left(\sqrt{\frac{2}{3}}\,y\right)^2}\,dy$$
$$=-\frac{1}{3}\,\frac{\arctan\left(\sqrt{\frac{2}{3}}\,y\right)}{\sqrt{\frac{2}{3}}}+c=-\frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{2}{3}}\cot x\right)+c$$

Additional examples

Substitution

Example 1. $I=\int\tan^6 x\,dx=?$

Substitution: $y=\tan x\implies x=x(t)=\arctan y$

$$x'(y)=\frac{dx}{dy}=\frac{1}{1+y^2}\implies dx=\frac{1}{1+y^2}\,dy$$

$$I = \int \tan^6 x \, dx = \int \frac{y^6}{y^2+1} \, dy$$

Here y^6 has to be divided by $y^2 + 1$. This can be done in several ways, for example:

$$\frac{y^6}{y^2+1} = \frac{y^4(y^2+1) - y^4}{y^2+1} = y^4 - \frac{y^4}{y^2+1} = y^4 - \frac{y^2(y^2+1) - y^2}{y^2+1} = y^4 - y^2 + \frac{y^2}{y^2+1} =$$

$$= y^4 - y^2 + \frac{y^2+1-1}{y^2+1} = y^4 - y^2 + 1 - \frac{1}{y^2+1}$$

or: $\dfrac{y^6}{y^2+1} = \dfrac{(y^6+1) - 1}{y^2+1} = \dfrac{(y^2+1)(y^4 - y^2 + 1) - 1}{y^2+1} = y^4 - y^2 + 1 - \dfrac{1}{y^2+1}$

or polynomial division can also be applied.

So $I = \displaystyle\int \frac{y^6}{y^2+1} \, dy = \int \left(y^4 - y^2 + 1 - \frac{1}{y^2+1} \right) dy = \frac{y^5}{5} - \frac{y^3}{3} + y - \arctan y + c =$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c$$

Example 2*. $I = \displaystyle\int \frac{1}{(x^2+1)^2} \, dx = ?$

Substitution: $x = x(t) = \tan t \quad \Longrightarrow \quad t = \arctan x$

$$x'(t) = \frac{dx}{dt} = \frac{1}{\cos^2 t} \quad \Longrightarrow \quad dx = \frac{1}{\cos^2 t} \, dt$$

$$I = \int \frac{1}{(x^2+1)^2} \, dx = \int \frac{1}{(\tan^2 t + 1)^2} \, \frac{1}{\cos^2 t} \, dt = \int \cos^4 t \, \frac{1}{\cos^2 t} \, dt = \int \cos^2 t \, dt =$$

$$\tan^2 t + 1 = \frac{1}{\cos^2 t}$$

$$= \int \frac{1 + \cos 2t}{2} \, dt = \frac{1}{2} t + \frac{1}{4} \sin 2t + c = \frac{1}{2} \arctan x + \frac{1}{2} \frac{x}{1+x^2} + c$$

$$\sin 2t = 2 \sin t \cos t = 2 \tan t \cos^2 t = \frac{2 \tan t}{1 + \tan^2 t} = \frac{2x}{1+x^2}$$

Example 3. $I = \displaystyle\int \frac{2x}{\sqrt{1-x^4}} \, dx = ?$

Substitution: $t = x^2 \implies x = x(t) = \sqrt{t} = t^{\frac{1}{2}}$

$$x'(t) = \frac{dx}{dt} = \frac{1}{2} t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}} \quad \Longrightarrow \quad dx = \frac{1}{2\sqrt{t}} \, dt$$

$$I = \int \frac{2x}{\sqrt{1-x^4}} \, dx = \int \frac{2\sqrt{t}}{\sqrt{1-t^2}} \cdot \frac{1}{2\sqrt{t}} \, dt = \int \frac{1}{\sqrt{1-t^2}} \, dt = \arcsin t + c = \arcsin(x^2) + c$$

Example 4. $I = \displaystyle\int \frac{1}{x\sqrt{x^2-1}} \, dx = ?$

Substitution: $t = \sqrt{x^2 - 1} \Longrightarrow t^2 = x^2 - 1 \Longrightarrow t^2 + 1 = x^2$

$$x = x(t) = \sqrt{t^2 + 1} = \left(t^2 + 1\right)^{\frac{1}{2}}$$

$$x'(t) = \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{1}{2}\left(t^2 + 1\right)^{-\frac{1}{2}} \cdot 2\,t = \frac{t}{\sqrt{t^2 + 1}} \Longrightarrow \mathrm{dx} = \frac{t}{\sqrt{t^2 + 1}}\,\mathrm{dt}$$

$$I = \int \frac{1}{x\sqrt{x^2 - 1}}\,\mathrm{dx} = \int \frac{1}{\sqrt{t^2 + 1} \cdot t} \cdot \frac{t}{\sqrt{t^2 + 1}}\,\mathrm{dt} = \int \frac{1}{t^2 + 1}\,\mathrm{dt} =$$

$$= \operatorname{arctg} t + c = \operatorname{arctg}\left(\sqrt{x^2 - 1}\right) + c$$