Calculus 1, Final exam 1, Part 1

18th December, 2023

Name: ________________________________ **Neptun code: ________________**

Part I: _________ **Part II.: _________** **Part III.: _________** **Sum: _________**

I. Definitions and theorems (15 x 3 points)

1. What does it mean that the sequence (a_n) is a Cauchy sequence?
2. State the Bolzano-Weierstrass theorem for number sequences.
3. State the ratio test for number series.
4. State Leibniz's theorem for number series.
5. What does it mean that the number $x \in \mathbb{R}$ is a limit point of the set $H \subset \mathbb{R}$?
6. What does it mean that $\lim_{x \to x_0} f(x) = +\infty$?
7. State the sequential criterion for continuity.
8. What does it mean that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous on an interval $J \subset \mathbb{R}$?
9. What does it mean that a function is convex? Write down the definition.
10. State Lagrange's mean value theorem.
11. State the L'Hospital's rule.
12. Give two sufficient conditions for a function to have a local minimum at the point x_0.
13. State Taylor's theorem with the remainder term.
14. State the integration-by-parts formula.
15. State the Newton-Leibniz formula.

II. Proof of a theorem (15 points)

Write down the statement of Bolzano's theorem (or intermediate value theorem) and prove it.

III. True or false? (15 x 3 points)

Indicate at each statement whether it is true or false and give a short explanation for your answer. The correct answer without an explanation is worth 1 point.

1. If $a_n > 1$ for all $n \in \mathbb{N}$, then $\lim_{n \to \infty} a_n^n = \infty$.

2. $\lim_{n \to \infty} a_n = L$ if and only if for any $\varepsilon > 0$ the sequence (a_n) has infinitely many terms closer to L than ε.

3. If the sequence (a_n) has no minimal term, then its limit cannot be $+\infty$.

4. If $a_n < b_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

5. a) If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} a_n^3$ is also convergent.

5. b) If x is a limit point of of $H \subset \mathbb{R}$, then x is a boundary point of H.

6. If $f:\mathbb{R}\longrightarrow\mathbb{R}$ is not continuous at x_0, then f doesn't have a finite limit at x_0.

7. The function $f(x)=\arctan\left(\frac{1}{x}\right)$ has a jump discontinuity at $x=0$.

8. The function $f(x)=2^{-x^2+4}-x\sqrt{x^2+5}$ has a real root in the interval $[0, 2]$.

9. If a function f is differentiable everywhere on $\mathbb{R}$ and $|f(9)-f(5)|\leq 2$, then $\left|f'(x)\right|\leq\frac{1}{2}$ for some $x\in[5, 9]$.

10. There exists a differentiable function $f:[a, b]\longrightarrow\mathbb{R}$ that has no maximum on $[a, b]$.

11. The function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is differentiable at x_0 if and only if f is differentiable at x_0 from the right and from the left.

12. Assume that f is at least two times differentiable on $\mathbb{R}$. If f has a local maximum at x_0 then $f'(x_0)=0$ and $f''(x_0)<0$.

13. The partial fraction decomposition of $f(x)=\frac{x+1}{(x-1)^3(x+2)^2}$ cannot contain the term $\frac{A}{(x+2)^3}$.

14. If the function $f:[a, b]\longrightarrow\mathbb{R}$ is Riemann-integrable, then it has an antiderivative.

15. There exists a function $f:[-1, 1]\longrightarrow\mathbb{R}$ whose integral function is $F(x)=\text{sgn}(x)$, $x\in[-1, 1]$.

Solutions

I. Definitions and theorems (15 x 3 points)

1. What does it mean that the sequence (a_n) is a Cauchy sequence?

Definition. (a_n) is a Cauchy sequence if for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that
if $n, m > N$ then $|a_n - a_m| < \varepsilon$.

2. State the Bolzano-Weierstrass theorem for number sequences.

Theorem. Every bounded sequence has a convergence subsequence.

3. State the ratio test for number series.

Theorem. Assume that $a_n > 0$. Then

(1) if $\limsup \frac{a_{n+1}}{a_n} < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent;

(2) if $\liminf \frac{a_{n+1}}{a_n} > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

4. State Leibniz's theorem for number series.

Theorem: Let (a_n) be a monotonically decreasing sequence of positive numbers such that $a_n \xrightarrow{n\to\infty} 0$.
Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots$ is convergent.

5. What does it mean that the number $x \in \mathbb{R}$ is a limit point of the set $H \subset \mathbb{R}$?

Definition. Let $H \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then x is a limit point of H, if for all $r > 0$: $(B(x, r) \setminus \{x\}) \cap H \neq \emptyset$
It means that any interval $(x - r, x + r)$ contains a point in H that is distinct from x.

6. What does it mean that $\lim_{x\to x_0} f(x) = +\infty$?

Definition. The limit of the function $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ is $+\infty$ if

(1) x_0 is a limit point of D_f $(x \in D_f{}')$

(2) for all $K > 0$ there exists $\delta(K) > 0$ such that
if $x \in D_f$ and $0 < |x - x_0| < \delta(K)$ then $f(x) > K$.

7. State the sequential criterion for continuity.

Theorem. The function $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $x_0 \in D_f$ if and only if
for all sequences $(x_n) \subset D_f$ for which $x_n \longrightarrow x_0$, $\lim_{n\to\infty} f(x_n) = f(x_0)$.

8. What does it mean that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous on an interval $J \subset \mathbb{R}$?

Definition. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous on the interval $J \subset \mathbb{R}$, if
$\forall\ \varepsilon > 0\ \ \exists\ \delta > 0$ such that $\forall\ x, y \in J$: $|x - y| < \delta \implies |f(x) - f(x)| < \varepsilon$.

9. What does it mean that a function is convex? Write down the definition.

Definition. The function f is concave on the interval $I \subset D_f$ if for all $x, y \in I$ and $t \in [0, 1]$
$f(t\,x + (1-t)\,y) \leq t\,f(x) + (1-t)\,f(y)$

Or:

Definition. Let $h_{a,b}(x)$ denote the the secant line passing through the points $(a, f(a))$ and $(b, f(b))$.
The function f is convex on the interval $I \subset D_f$ if for all $\forall\, a, b \in I$ and $a < x < b \implies f(x) \leq h_{a,b}(x)$,
that is, the secant lines of f always lie above the graph of f.

10. State Lagrange's mean value theorem.

Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b).

Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

11. State the L'Hospital's rule.

Theorem.
Assume that $a \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, I is a neighbourhood of a, the functions f and g are differentiable on $I \setminus \{a\}$ and $g(x) \neq 0$, $g'(x) \neq 0$ for all $x \in I \setminus \{a\}$. Assume moreover that

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} |\, f(x)\,| = \lim_{x\to a} |\, g(x)\,| = \infty.$$

If $\exists \lim_{x\to a} \frac{f'(x)}{g'(x)} = b \in \overline{\mathbb{R}}$ then $\exists \lim_{x\to a} \frac{f(x)}{g(x)} = b$.

12. Give two sufficient conditions for a function to have a local minimum at the point x_0.

Theorems.
1) Assume that f is differentiable at $x_0 \in \operatorname{int} D_f$.
   If $f'(x_0) = 0$ and f' changes sign from negative to positive at x_0, then f has a local minimum at x_0.
2) Assume that f is twice differentiable at $x_0 \in \operatorname{int} D_f$.
   If $f'(x_0) = 0$ and $f''(x_0) > 0$ then f has a local minimum at x_0.

13. State Taylor's theorem with the remainder term.

Theorem (Taylor’s theorem). Assume that f is at least $(n+1)$ times differentiable on the interval $(x_0 - \delta,\ x_0 + \delta)$ and $x \in (x_0 - \delta,\ x_0 + \delta)$. Then there exists a number ξ between x and x_0 (that is, $x_0 < \xi < x$ or $x < \xi < x_0$) such that

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x - x_0)^{n+1}.$$

This expression is called the Lagrange form of the remainder term.

14. State the integration-by-parts formula.

Theorem. Assume that f and g are differentiable on the interval I and $f \cdot g'$ has an antiderivative on I. Then $f' \cdot g$ also has an antiderivative here and

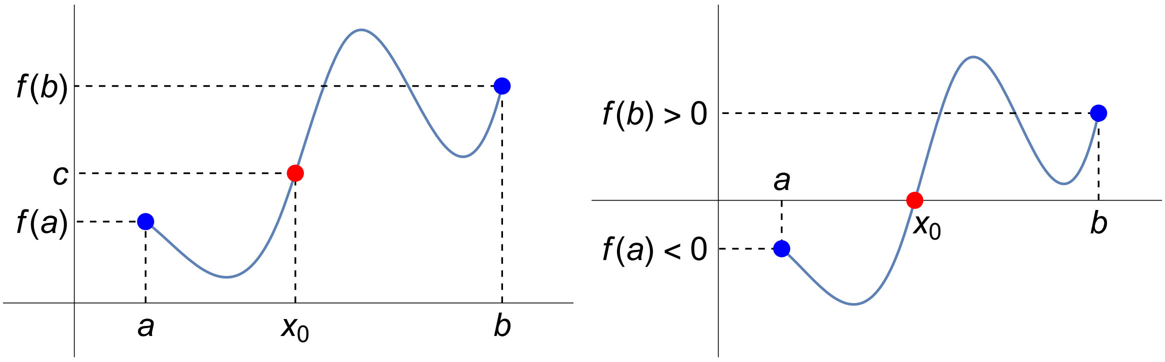
$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx$$

15. State the Newton-Leibniz formula.

Theorem. If $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable and $F : [a, b] \longrightarrow \mathbb{R}$ is an antiderivative of f, that is, $F'(x) = f(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\, dx = F(b) - F(a) = [F(x)]_a^b$.

II. Proof of a theorem (15 points)

Theorem (Intermediate value theorem or Bolzano's theorem).
Assume that f is continuous on $[a, b]$, $f(a) \neq f(b)$ and $f(a) < c < f(b)$ or $f(b) < c < f(a)$.
Then there exists $x_0 \in (a, b)$ such that $f(x_0) = c$.



Proof. We prove the case $f(a) < c < f(b)$. The point x_0 can be found with an interval halving method (bisection method).

1st step: Consider the midpoint $\frac{a+b}{2}$ of the interval $[a, b]$. There are three cases:

If $f\left(\frac{a+b}{2}\right) > c \implies a_1 := a,\ b_1 := \frac{a+b}{2}$

If $f\left(\frac{a+b}{2}\right) < c \implies a_1 := \frac{a+b}{2},\ b_1 := b$

If $f\left(\frac{a+b}{2}\right) = c \implies x_0 := \frac{a+b}{2}$

2nd step: Consider the midpoint $\frac{a_1+b_1}{2}$ of the interval $[a_1, b_1]$. There are again three cases:

If $f\left(\frac{a_1+b_1}{2}\right) > c \implies a_2 := a_1,\ b_2 := \frac{a_1+b_1}{2}$

If $f\left(\frac{a_1+b_1}{2}\right) < c \implies a_2 := \frac{a_1+b_1}{2},\ b_2 := b_1$

If $f\left(\frac{a_1+b_1}{2}\right) = c \implies x_0 := \frac{a_1+b_1}{2}$

Continuing the above procedure, we either reach x_0 in one of the steps, or we define the sequences (a_n) and (b_n) such that

$$[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \ldots \supset [a_n, b_n] \supset [a_{n+1}, b_{n+1}] \supset \ldots,$$

and

$$b_1 - a_1 = \frac{b-a}{2},\ b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b-a}{2^2},\ \ldots,\ b_n - a_n = \frac{b-a}{2^n},\ \ldots$$

From this it follows that $\lim_{n\to\infty}(b_n - a_n) = 0$, so by the Cantor axiom there exists a unique element $x_0 \in [a, b]$ such that $\bigcap_{n=1}^{\infty}[a_n, b_n] = \{x_0\}$.

Then $a_n \longrightarrow x_0$, $b_n \longrightarrow x_0$, so by the continuity of f we have that $\lim_{n\to\infty} f(a_n) = f(x_0) = \lim_{n\to\infty} f(b_n)$, and since $f(a_n) \le c \le f(b_n)$, it follows that $f(x_0) = c$.

III. True or false? (15 x 3 points)

1. If $a_n > 1$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n^n = \infty$.

False. For example, $a_n = 1 + \frac{1}{n} > 1$ and $\lim_{n\to\infty} a_n^n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$.

2. $\lim_{n\to\infty} a_n = L$ if and only if for any $\varepsilon > 0$ the sequence (a_n) has infinitely many terms closer to L than ε.

False. For example, if $a_n = (-1)^n$ and $\varepsilon = 1$, then (a_n) has infinitely many terms (the terms with an even index) that are closer to $L = 1$ than ε (that is, $0 < a_{2n} < 2$), but (a_n) is divergent, so $L = 1$ is not the limit.

3. If the sequence (a_n) has no minimal term, then its limit cannot be $+\infty$.

True. The contrapositive of this statement is: if $\lim_{n\to\infty} a_n = +\infty$, then the sequence (a_n) has a minimal term. This statement is true, since $\lim_{n\to\infty} a_n = +\infty$ means that for all $K > 0$, the sequence has only finitely many terms that are less than K. Among finitely many terms there is a minimal term. Since the contrapositive of the statement is true, then the original statement is also true.

4. If $a_n < b_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

False. For example, if $a_n = -1 < 0 = b_n$, then $\sum_{n=1}^{\infty} a_n$ is divergent but $\sum_{n=1}^{\infty} b_n$ is convergent.

The implication is only true if $a_n \ge 0$.

5. a) If $a_n \ge 0$ and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} a_n^3$ is also convergent.

True. If $\sum_{n=1}^{\infty} a_n$ converges, then by the nth term test $a_n \longrightarrow 0$. Then by the definition of the limit, there exists $n \in \mathbb{N}$ such that for all $n > \mathbb{N}$ we have $0 \le a_n < 1$. From this it follows that $0 \le a_n^3 \le a_n < 1$ also holds. Since $\sum_{n=1}^{\infty} a_n$ converges, then by the comparison test $\sum_{n=1}^{\infty} a_n^3$ also converges.

5. b) If x is a limit point of of $H \subset \mathbb{R}$, then x is a boundary point of H.

False. For example, $x = 1$ is both a limit point and interior point of $H = (0, 2)$, but not a boundary point.

6. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is not continuous at x_0, then f doesn't have a finite limit at x_0.

False. If f has a removable discontinuity at x_0, then $\exists \lim_{x \to x_0} f(x) \in \mathbb{R}$.

7. The function $f(x) = \arctan\left(\frac{1}{x}\right)$ has a jump discontinuity at $x = 0$.

True. $\lim_{x \to 0+0} \frac{1}{x} = +\infty$ and $\lim_{x \to 0-0} \frac{1}{x} = -\infty \Longrightarrow \lim_{x \to 0+0} \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$ and $\lim_{x \to 0-0} \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}$, so f has a jump discontinuity at $x = 0$.

8. The function $f(x) = 2^{-x^2+4} - x\sqrt{x^2+5}$ has a real root in the interval $[0, 2]$.

True. $f(0) = 2^4 - 0 = 16 > 0$ and $f(2) = 2^0 - 2 \cdot 3 = -5 < 0$, so by Bolzano's theorem f has a real root in the interval $[0, 2]$.

9. If a function f is differentiable everywhere on $\mathbb{R}$ and $|f(9) - f(5)| \leq 2$, then $\left|f'(x)\right| \leq \frac{1}{2}$ for some $x \in [5, 9]$.

True. By Lagrange's theorem there exists $c \in (5, 9)$, such that $f'(c) = \frac{f(9) - f(5)}{9 - 5} \Longrightarrow$

$|f'(c)| = \frac{|f(9) - f(5)|}{4} \leq \frac{2}{4} = \frac{1}{2}$.

10. There exists a differentiable function $f : [a, b] \longrightarrow \mathbb{R}$ that has no maximum on $[a, b]$.

False. Since f is differentiable, then f is continuous on $[a, b]$, so by Weierstrass' extreme value theorem f has a maximum (and minimum) on $[a, b]$.

11. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at x_0 if and only if f is differentiable at x_0 from the right and from the left.

False. For example, $f(x) = |x|$ is differentiable at $x = 0$ from the right and from the left ($f_+'(0) = 1$, $f_-'(0) = 1$), but since the one-sided derivatives are not equal, then f is not differentiable at $x = 0$.

12. Assume that f is at least two times differentiable on $\mathbb{R}$. If f has a local maximum at x_0 then $f'(x_0) = 0$ and $f''(x_0) < 0$.

False. For example, if $f(x) = -x^4$, then f has a local maximum at $x_0 = 0$, but $f'(0) = f''(0) = 0$.

13. The partial fraction decomposition of $f(x) = \frac{x+1}{(x-1)^3 (x+2)^2}$ cannot contain the term $\frac{A}{(x+2)^3}$.

True. The partial fraction decomposition is $f(x) = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+2} + \frac{E}{(x+2)^2}$.

14. If the function $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann-integrable, then it has an antiderivative.

False. For example, if $f(x) = \text{sgn}(x)$, then the integral $\int_{-1}^{1} \text{sgn}(x)\,dx$ exists, since f is continuous except one point. However, by Darboux's theorem, f doesn't have an antiderivative, since f has a jump discontinuity.

15. There exists a function $f : [-1, 1] \longrightarrow \mathbb{R}$ whose integral function is $F(x) = \text{sgn}(x),\ x \in [-1, 1]$.

False. The integral function of f is Lipschitz continuous on $[-1, 1]$, so it is continuous.
However, $F(x) = \text{sgn}(x)$ has a jump discontinuity at $x = 0$, therefore it cannot be an integral function.