Calculus 1, Final exam 1, Part 2

18th December, 2023

Name:_______________________________ Neptun code:__________

1.:_____ 2.:_____ 3.:_____ 4.:_____ 5.:_____ 6.:_____ 7.:_____ 8.:_____ Sum:_____

1. (10 points) Calculate the following limit: $\lim_{x\to 0} \dfrac{x \arctan(3x)}{\cos(2x) - 1}$

2. (5+5+5 points) Calculate the derivatives of the following functions:

a) $f(x) = 2\sqrt{x+1} - 2\arctan\left(\sqrt{x+1}\right)$ b) $g(x) = \tan\left(\sqrt{\dfrac{e^{x^2}+2}{x^2+1}}\right)$

c) $h(x) = \int_0^{\sqrt{x}} \sqrt{\sin^2 t + 1}\, dt$

3. (10 points) Give Taylor polynomial of order 3 of the function $f(x) = x^3 - 6x^2 + 7 + \cos(2x)$ around the point $x_0 = 0$. Estimate the error if the value of $f(0.2)$ is approximated by $T_3(0.2)$.

4. (15 points) Analyze the following function and sketch its graph: $f(x) = x\, e^{-2x^2}$.

5. (10+10 points) Calculate the following integrals:

a) $I_1 = \int \arcsin(2x)\, dx$ b) $I_2 = \int \ln\left(\sqrt{x} + 1\right) dx$ (substitution: $t = \sqrt{x}$)

6. (10+10 points) Calculate the following integrals:

a) $I_3 = \int \dfrac{x+4}{(x-1)\left(x^2+4\right)}\, dx$ b) $I_4 = \int \dfrac{e^{2x} - e^x}{e^{2x} + 5e^x + 6}\, dx$ (substitution: $t = e^x$)

7. (10 points) Consider the function $f(x) = \dfrac{1}{\sqrt{x}\, \ln x}$ on the interval $x \in \left[e^2, e^3\right]$.

Rotate it around the x-axis and find the volume of the arising body.

8.* (10 points - BONUS) The radius of the base circle of a right circular cone is 2 meters, and its height is 5 meters. Determine the dimensions of the cylinder with the maximum volume that can be inscribed in the cone.

Solutions

1. (10 points) Calculate the following limit: $\lim_{x\to 0} \frac{x \arctan(3x)}{\cos(2x)-1}$

Solution. The limit has the form $\frac{0}{0}$, so the L'Hospital's rule can be applied:

$$\lim_{x\to 0} \frac{x \arctan(3x)}{\cos(2x)-1} \overset{\frac{0}{0}, L'H}{=} \lim_{x\to 0} \frac{\arctan(3x) + \frac{3x}{1+9x^2}}{-\sin(2x)\cdot 2} \ \textbf{(4p)} \overset{\frac{0}{0}, L'H}{=} \lim_{x\to 0} \frac{\frac{3}{1+9x^2} + \frac{3(1+9x^2)-3x\cdot 18x}{(1+9x^2)^2}}{-\cos(2x)\cdot 4} \ \textbf{(4p)}$$

$$= \frac{3+3}{(-1)\cdot 4} = -\frac{3}{2} \ \textbf{(2p)}$$

2. (5+5+5 points) Calculate the derivatives of the following functions:

a) $f(x) = 2\sqrt{x+1} - 2\arctan\left(\sqrt{x+1}\right)$ b) $g(x) = \tan\left(\sqrt{\frac{e^{x^2}+2}{x^2+1}}\right)$

c) $h(x) = \int_0^{\sqrt{x}} \sqrt{\sin^2 t + 1}\, dt$

Solution. a) $f'(x) = 2\cdot\frac{1}{2}(x+1)^{-\frac{1}{2}} - 2\cdot\frac{1}{1+\left(\sqrt{x+1}\right)^2}\cdot\frac{1}{2}(x+1)^{-\frac{1}{2}}$ **(5p)**

$$= \frac{1}{\sqrt{x+1}} - \frac{1}{x+2}\cdot\frac{1}{\sqrt{x+1}} = \frac{x+2-1}{(x+2)\sqrt{x+1}} = \frac{x+1}{(x+2)\sqrt{x+1}} = \frac{\sqrt{x+1}}{x+2}$$

b) $g'(x) = \frac{1}{\cos^2\left(\sqrt{\frac{e^{x^2}+2}{x^2+1}}\right)}\cdot\frac{1}{2}\left(\frac{e^{x^2}+2}{x^2+1}\right)^{-\frac{1}{2}}\cdot\frac{e^{x^2}\cdot 2x(x^2+1)-\left(e^{x^2}+2\right)\cdot 2x}{(x^2)^2}$ **(5p)**

c) $h(x) = \int_0^{\sqrt{x}} \sqrt{\sin^2 t + 1}\, dt = F\left(\sqrt{x}\right)$ where $F(x) = \int_0^x \sqrt{\sin^2 t + 1}\, dt$

Since the integrand $\sqrt{\sin^2 t + 1}$ is a continuous function of t then F is differentiable and $F'(x) = \sqrt{\sin^2 x + 1}$

$\Longrightarrow h'(x) = F'\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}} = \sqrt{\sin^2\left(\sqrt{x}\right)+1}\cdot\frac{1}{2\sqrt{x}}$ **(5p)**

3. (10 points) Give Taylor polynomial of order 3 of the function $f(x) = x^3 - 6x^2 + 7 + \cos(2x)$ around the point $x_0 = 0$. Estimate the error if the value of $f(0.2)$ is approximated by $T_3(0.2)$.

Solution.

$f(x) = x^3 - 6x^2 + 7 + \cos(2x) \Longrightarrow f(0) = 8$

$f'(x) = 3x^2 - 12x - 2\sin(2x) \Longrightarrow f'(0) = 0$

$f''(x) = 6x - 12 - 4\cos(2x) \Longrightarrow f''(0) = -16$

$f'''(x) = 6 + 8\sin(2x) \quad \Longrightarrow \quad f'''(0) = 6$ **(3p)**

$f^{(4)}(x) = 16\cos(2x)$

The Taylor polynomial at $x_0 = 0$:

$$T_n(x) = 8 + 0\cdot(x-0) + \frac{-16}{2!}(x-0)^2 + \frac{6}{3!}(x-0)^3 = 8 - 8x^2 + x^3 \quad \textbf{(3p)}$$

Lagrange remainder term: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$, where $n = 3$, $x_0 = 0$, $x = 0.2$, $0 < \xi < 0.2$

Taylor's theorem: $f(x) = T_n(x) + R_n(x)$

The error for the approximation $f(x) \approx T_3(x)$ can be estimate from above:

$$|E| = |f(x) - T_3(x)| = |R_3(x)| = \left|\frac{f^{(4)}(\xi)}{4!}(0.2-0)^4\right| = \left|\frac{16\cos(2\xi)}{4!}\cdot 0.2^4\right|$$

$$= \frac{16\cdot|\cos(2\xi)|}{4!}\cdot 0.2^4 \le \frac{16\cdot 1}{4!}\cdot 0.2^4 \quad \textbf{(4p)} \quad \approx 0.00106667$$

Remark. Comparison of the numerical values:

$f(0.2) \approx 7.68906$

$T_3(0.2) = 8 - 8\cdot 0.2^2 + 0.2^3 \approx 7.688$

4. (15 points) Analyze the following function and sketch its graph: $f(x) = x\, e^{-2x^2}$.

Solution.

$D_f = \mathbb{R}$; $f(x) = 0 \Longleftrightarrow x = 0$; f is odd

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\frac{x}{e^{2x^2}} \overset{L'H}{=} \lim_{x\to\pm\infty}\frac{1}{e^{2x^2}\cdot 4x} = 0 \quad \textbf{(2p)}$$

$$f'(x) = e^{-2x^2}(1 - 4x^2) = 0 \Longleftrightarrow x = \pm\frac{1}{2} \quad \textbf{(2p)}$$

x	$x<-\frac{1}{2}$	$x=-\frac{1}{2}$	$-\frac{1}{2}<x<\frac{1}{2}$	$x=\frac{1}{2}$	$x>\frac{1}{2}$
f'	–	0	+	0	–
f	↘	min: $-\frac{1}{2\sqrt{e}} \approx -0.303$	↗	max: $\frac{1}{2\sqrt{e}} \approx 0.303$	↘

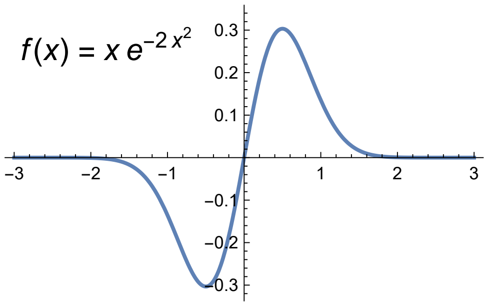
(3p)

$$f''(x) = 4e^{-2x^2}x(-3 + 4x^2) = 0 \Longleftrightarrow x = 0 \text{ or } x = \pm\frac{\sqrt{3}}{2} \quad \textbf{(2p)}$$

x	$x<-\frac{\sqrt{3}}{2}$	$x=-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}<x<0$	x=0	$0<x<\frac{\sqrt{3}}{2}$	$x=\frac{\sqrt{3}}{2}$	$x>\frac{\sqrt{3}}{2}$
f''	–	0	+	0	–	0	+
f	∩	infl: ≈ −0.193	∪	infl: 0	∩	infl: ≈ 0.193	∪

(3p)

The graph of f **(3p)**



$\Longrightarrow$ the range of f is $\left[-\frac{1}{2\sqrt{e}}, \frac{1}{2\sqrt{e}}\right]$

5. (10+10 points) Calculate the following integrals:

a) $I_1 = \int \arcsin(2x)\,dx$ b) $I_2 = \int \ln\left(\sqrt{x}+1\right) dx$ (substitution: $t = \sqrt{x}$)

Solution. a) $I_1 = \int \arcsin(2x)\,dx = ?$

With integration by parts:

$$\int \arcsin(2x)\,dx = \int 1\cdot\arcsin(2x)\,dx = x\cdot\arcsin(2x) - \int x\cdot\frac{2}{\sqrt{1-4x^2}}\,dx \quad \textbf{(3p)}$$

$$= x\cdot\arcsin(2x) + \frac{1}{4}\int(-8x)\cdot\left(1-4x^2\right)^{-\frac{1}{2}}\,dx = \quad \textbf{(4p)}$$

$$= x\cdot\arcsin(2x) + \frac{1}{4}\,\frac{\left(1-4x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = x\arcsin(2x) + \frac{1}{2}\cdot\sqrt{1-4x^2} + c \quad \textbf{(3p)}$$

b) $I_2 = \int \ln\left(\sqrt{x}+1\right) dx = ?$ Substitution: $t = \sqrt{x} \Longrightarrow x = x(t) = t^2 \Longrightarrow x'(t) = \frac{dx}{dt} = 2t \Longrightarrow dx = 2t\,dt$

$\Longrightarrow I_2 = \int \ln(t+1)\cdot 2t\,dt =$ **(3p)**

With integration by parts: $f'(t) = 2t \Longrightarrow f(t) = t^2$

$g(t) = \ln(t+1) \Longrightarrow g'(t) = \frac{1}{t+1}$

$$I_2 = t^2\ln(t+1) - \int t^2\cdot\frac{1}{t+1}\,dt \quad \textbf{(3p)} \quad = t^2\ln(t+1) - \int\frac{t^2-1+1}{t+1}\,dt = t^2\ln(t+1) - \int\frac{(t-1)(t+1)+1}{t+1}\,dt =$$

$$= t^2\ln(t+1) - \int\left(t-1+\frac{1}{t+1}\right)dt = t^2\ln(t+1) - \frac{t^2}{2} + t - \ln|t+1| + c \quad \textbf{(3p)}$$

$$= x\ln\left(\sqrt{x}+1\right) - \frac{x}{2} + \sqrt{x} - \ln\left(\sqrt{x}+1\right) + c \quad \textbf{(1p)}$$

6. (10+10 points) Calculate the following integrals:

a) $I_3 = \int \frac{x+4}{(x-1)\left(x^2+4\right)}\,dx$ b) $I_4 = \int \frac{e^{2x}-e^x}{e^{2x}+5e^x+6}\,dx$ (substitution: $t = e^x$)

Solution. a) We use partial fraction decomposition:

$\frac{x+4}{(x-1)\left(x^2+4\right)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$ **(2p)** Multiplying by $(x-1)\left(x^2+4\right)$ we get:

$$x+4=A(x^2+4)+(x-1)(Bx+C)$$

$$\begin{aligned}
x=1 &\Longrightarrow \quad 5=5A+0 &&\Longrightarrow A=1\\
x=0 &\Longrightarrow \quad 4=4A-C &&\Longrightarrow C=0 \quad \textbf{(3p)}\\
x=2 &\Longrightarrow \quad 6=8A+2B+C &&\Longrightarrow B=3-4A \Longrightarrow B=-1
\end{aligned}$$

$$\Longrightarrow I_3=\int\frac{x+4}{(x-1)(x^2+4)}\,dx=\int\left(\frac{1}{x-1}-\frac{x}{x^2+4}\right)dx=\int\left(\frac{1}{x-1}-\frac{1}{2}\frac{2x}{x^2+4}\right)dx=$$

$$=\ln|x-1|-\frac{1}{2}\ln(x^2+4)+c \quad \textbf{(5p)}$$

b) $I_4=\int\frac{e^{2x}-e^x}{e^{2x}+5e^x+6}\,dx=?$ (substitution: $t=e^x$)

Substitution: $t=e^x \Longrightarrow x=x(t)=\ln t \Longrightarrow x'(t)=\frac{dx}{dt}=\frac{1}{t} \Longrightarrow dx=\frac{1}{t}\,dt$

$$\Longrightarrow I_4=\int\frac{e^{2x}-e^x}{e^{2x}+5e^x+6}\,dx=\int\frac{t^2-t}{t^2+5t+6}\cdot\frac{1}{t}\,dt=\int\frac{t-1}{(t+2)(t+3)}\,dt \quad \textbf{(4p)}$$

Partial fraction decomposition: $\frac{t-1}{(t+2)(t+3)}=\frac{A}{t+2}+\frac{B}{t+3}$

$$\begin{aligned}
&\Longrightarrow \quad t-1=A(t+3)+B(t+2)\\
t=-2 &\Longrightarrow \quad -3=A+0 \Longrightarrow A=-3\\
t=-3 &\Longrightarrow \quad -4=0-B \Longrightarrow B=4 \quad \textbf{(3p)}
\end{aligned}$$

$$\Longrightarrow I_4=\int\left(\frac{-3}{t+2}+\frac{4}{t+3}\right)dt=-3\ln|t+2|+4\ln|t+3|+c=-3\ln(e^x+2)+4\ln(e^x+3)+c \quad \textbf{(3p)}$$

7. (10 points) Consider the function $f(x)=\frac{1}{\sqrt{x}\,\ln x}$ on the interval $x\in[e^2, e^3]$.

Rotate it around the x-axis and find the volume of the arising body.

Solution. The volume is $V=\pi\int_{e^2}^{e^3}f^2(x)\,dx=\pi\int_{e^2}^{e^3}\frac{1}{x\ln^2 x}\,dx$ **(3p)**

$$=\pi\int_{e^2}^{e^3}\frac{1}{x}\cdot(\ln x)^{-2}\,dx=\pi\left[\frac{(\ln x)^{-1}}{-1}\right]_{e^2}^{e^3} \quad \textbf{(4p)} \quad =\pi\left[-\frac{1}{\ln x}\right]_{e^2}^{e^3}$$

$$=\pi\left[-\frac{1}{\ln e^3}+\frac{1}{\ln e^2}\right] \quad \textbf{(2p)} \quad =\pi\left(-\frac{1}{3}+\frac{1}{2}\right)=\frac{\pi}{6} \quad \textbf{(1p)}$$

8.* (10 points - BONUS) The radius of the base circle of a right circular cone is 2 meters, and its height is 5 meters. Determine the dimensions of the cylinder with the maximum volume that can be inscribed in the cone.

Solution. The height of the cone is $m=5$ and the radius of the base circle is $r=2$.
Let x and y respectively denote the radius of the base circle and the height of the cylinder and let

α denote the angle formed by the slant height of the cone with the plane of its base.
Then $\tan\alpha = \frac{m}{r} = \frac{y}{r-x}$.
The volume of the cylinder that we want to maximize is
$V(x) = \pi x^2 y = \pi\left(\frac{5}{2} x^2 (2-x)\right) = \pi\left(-\frac{5}{2} x^3 + 5 x^2\right)$, where $0 < x < 2$.
Then $V'(x) = \pi\left(-\frac{15}{2} x^2 + 10 x\right) = 0$, from where $x = \frac{4}{3}$ (since $x > 0$).
At this point the function has a maximum, since
$V''(x) = \pi(-15 x + 10)$, so $V''\left(\frac{4}{3}\right) = -10\pi < 0$.
The radius of the base circle and the height of the cylinder with maximum volume are
$x = \frac{4}{3}$ and $y = \frac{5}{3}$ meters, respectively.