Calculus 1, Final exam 2, Part 1

12th January, 2024

Name: ______________________________ Neptun code: _______________

Part I: ________ Part II.: ________ Part III.: ________ Sum: ________

I. Definitions and theorems (15 x 3 points)

1. What is the statement of the sandwich theorem for number sequences?
2. Define the limit point and the limes superior of the sequence (a_n).
3. State the root test for number series.
4. State Leibniz's theorem for alternating series.
5. What does it mean that
   a) $\lim\limits_{x\to\infty} f(x) = A \in \mathbb{R}$? b) $\lim\limits_{x\to\infty} f(x) \neq A \in \mathbb{R}$?
6. What does it mean that a function f has a removable discontinuity at the point $x_0 \in \mathbb{R}$?
7. State the intermediate value theorem or Bolzano's theorem.
8. What does it mean that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous on an interval $J \subset \mathbb{R}$?
9. What does it mean that a function is concave? Write down the definition.
10. State Rolle's theorem.
11. State Darboux's theorem.
12. Give two sufficient conditions for a function to have an inflection point at x_0.
13. State Taylor's theorem with the remainder term.
14. Give two sufficient conditions for a function $f : [a,\ b] \longrightarrow \mathbb{R}$ to be Riemann integrable.
15. State the second fundamental theorem of calculus.

II. Proof of a theorem (15 points)

Write down the statement of the Newton-Leibniz formula and prove it.

III. True or false? (16 x 3 points)

Indicate at each statement whether it is true or false and give a short explanation for your answer. The correct answer without an explanation is worth 1 point.

1. $\lim\limits_{n\to\infty} a_n = L$ if and only if there exists $\varepsilon > 0$ such that the sequence (a_n) has infinitely many terms closer to L than ε.

2. The sequence $(a_n \cdot b_n)$ is convergent if and only if both (a_n) and (b_n) are convergent.

3. The sequence $a_n = \cos n$ has a convergent subsequence.

4. If $a_n \longrightarrow 0$, then $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

5. If $a_n > 0$ for all $n \in \mathbb{N}$ and the sequence (a_n) is monotonically decreasing, then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent.

6. If the set $H \subset \mathbb{R}$ contains all of its limit points, then it is closed.

7. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and assume that for all sequences (x_n) with non-zero terms for which $\lim_{n\to\infty} x_n = 0$, we have $\lim_{n\to\infty} f(x_n) = A$. Then $\lim_{x\to 0} f(x) = A$.

8. The function $f(x) = \frac{1}{\ln(x^2)}$ has a jump discontinuity at $x = 0$.

9. The equation $x^5 = 10 x^2 - 3$ has a real solution in the interval $[0, 1]$.

10. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at x_0, then the limit $\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists and is finite.

11. The function $f(x) = e^x + \arctan(x)$ is invertible on $\mathbb{R}$.

12. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$. Then f has both a minimum and a maximum on $[a, b]$ only if f is differentiable on (a, b).

13. Assume that f is at least two times differentiable on $\mathbb{R}$. If $f''(x_0) = 0$, then f has an inflection point at x_0.

14. The partial fraction decomposition of $f(x) = \frac{x + 6}{(x + 1)(x^4 - 1)}$ contains the term $\frac{A x + B}{x^2 + 1}$.

15. The function $f(x) = \operatorname{sgn}(x)$ has an antiderivative on $[-1, 1]$.

16. If $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and has a finite number of discontinuities, then f is Riemann-integrable on $[a, b]$.

Solutions

I. Definitions and theorems (15 x 3 points)

1. What is the statement of the sandwich theorem for number sequences?

Theorem. If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$, $c_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ and $a_n \le b_n \le c_n$ for all $n > N$, then $b_n \xrightarrow{n\to\infty} A \in \mathbb{R}$.

2. Define the limit point and the limes superior of the sequence (a_n).

Definition. $A \in \mathbb{R} \cup \{\infty, -\infty\}$ is called a limit point or accumulation point of (a_n) if any neighbourhood of A contains infinitely many terms of (a_n).
Or equivalently there exists a subsequence (a_{n_k}) such that $a_{n_k} \xrightarrow{n\to\infty} A$.

Definition.
- If the set of limit points of (a_n) is bounded above, then its supremum is called the limes superior of (a_n) (notation: $\limsup a_n$).
- If (a_n) is not bounded above, then we define $\limsup a_n = \infty$.

3. State the root test for number series.

Theorem. Assume that $a_n > 0$ and $\limsup \sqrt[n]{a_n} = R$. Then

(1) if $R < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent;

(2) if $R > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

4. State Leibniz's theorem for alternating series.

Theorem: Let (a_n) be a monotonically decreasing sequence of positive numbers such that $a_n \xrightarrow{n\to\infty} 0$.
Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots$ is convergent.

5. What does it mean that
a) $\lim_{x\to\infty} f(x) = A \in \mathbb{R}$? b) $\lim_{x\to\infty} f(x) \ne A \in \mathbb{R}$?

a) Definition: $\lim_{x\to\infty} f(x) = A \in \mathbb{R} \iff$ For all $\varepsilon > 0$ there exists $P(\varepsilon) > 0$ such that for all $x > P(\varepsilon)$ we have $|f(x) - A| < \varepsilon$.
b) Negation of the definition:
$\lim_{x\to\infty} f(x) \ne A \in \mathbb{R} \iff$ There exists $\varepsilon > 0$ such that for all $P(\varepsilon) > 0$ there exists $x > P(\varepsilon)$, for which $|f(x) - A| \ge \varepsilon$.

6. What does it mean that a function f has a removable discontinuity at the point $x_0 \in \mathbb{R}$?

Definition. f has a removable discontinuity at x_0 if $\exists \lim_{x\to x_0} f(x) \in \mathbb{R}$ but $\lim_{x\to x_0} f(x) \ne f(x_0)$ or $f(x_0)$ is not defined.

7. State the intermediate value theorem or Bolzano's theorem.

Theorem. Assume that f is continuous on $[a, b]$, $f(a) \neq f(b)$ and $f(a) < c < f(b)$ or $f(b) < c < f(a)$.
Then there exists $x_0 \in (a, b)$ such that $f(x_0) = c$.

8. What does it mean that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous on an interval $J \subset \mathbb{R}$?

Definition. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous on the interval $J \subset \mathbb{R}$, if

$$\forall\ \varepsilon > 0 \quad \exists\ \delta > 0 \ \text{ such that } \forall\, x, y \in J : \quad |\, x - y \,| < \delta \implies |\, f(x) - f(x) \,| < \varepsilon.$$

9. What does it mean that a function is concave? Write down the definition.

Definition. The function f is concave on the interval $I \subset D_f$ if for all $x, y \in I$ and $t \in [0, 1]$
$f(t\,x + (1 - t)\,y) \geq t\,f(x) + (1 - t)\,f(y)$

Or:

Definition. Let $h_{a,b}(x)$ denote the the secant line passing through the points $(a, f(a))$ and $(b, f(b))$.
The function f is concave on the interval $I \subset D_f$ if for all $\forall\, a, b \in I$ and $a < x < b \implies f(x) \geq h_{a,b}(x)$,
that is, the secant lines of f always lie below the graph of f.

10. State Rolle's theorem.

Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b)
and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

11. State Darboux's theorem.

Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is differentiable and $f'(a) < y < f'(b)$ or $f'(b) < y < f'(a)$.
Then there exists $c \in (a, b)$ such that $f'(c) = y$.

12. Give two sufficient conditions for a function to have an inflection point at x_0.

Theorem (Sufficient condition for an inflection point, second derivative test).
If f is twice differentiable in a neighbourhood of x_0,
$f''(x_0) = 0$ and f'' changes sign at x_0,
then f has an inflection point at x_0.

Theorem (Sufficient condition for an inflection point, third derivative test).
If f is three times differentiable in a neighbourhood of x_0,
$f''(x_0) = 0$ and $f'''(x_0) \neq 0$,
then f has an inflection point at x_0.

13. State Taylor's theorem with the remainder term.

Theorem (Taylor’s theorem). Assume that f is at least $(n + 1)$ times differentiable
on the interval $(x_0 - \delta,\ x_0 + \delta)$ and $x \in (x_0 - \delta,\ x_0 + \delta)$. Then there exists a number ξ
between x and x_0 (that is, $x_0 < \xi < x$ or $x < \xi < x_0$) such that

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

This expression is called the Lagrange form of the remainder term.

14. Give two sufficient conditions for a function $f : [a, b] \longrightarrow \mathbb{R}$ to be Riemann integrable.

Theorems:
1) If f is monotonic and bounded on $[a, b]$ then f is Riemann integrable on $[a, b]$.
2) If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous then f is Riemann integrable on $[a, b]$.
3) If $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and continuous except finitely many points then f is Riemann integrable on $[a, b]$.
Any two conditions are suitable.

15. State the second fundamental theorem of calculus.

Theorem. Assume that f is Riemann integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$.

Then
1. F is Lipschitz continuous on $[a, b]$.
2. If f is continuous at $x_0 \in [a, b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

II. Proof of a theorem (15 points)

Theorem (First fundamental theorem of calculus, Newton-Leibniz formula).
If $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable and $F : [a, b] \longrightarrow \mathbb{R}$ is an antiderivative of f, that is, $F'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b$$

Proof. Let (P_n) be a partition sequence of $[a, b]$ such that $\lim_{n\to\infty} \Delta P_n = 0$.

For all $k \in \{1, 2, \ldots, n\}$, F is continuous on $[x_{k-1}, x_k]$ and differentiable on (x_{k-1}, x_k), so by Lagrange's mean value theorem there exists $x_{k-1} < c_k < x_k$ such that

$$\frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = F'(c_k) = f(c_k) \implies F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1})$$

$$\implies F(b) - F(a) = (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + \ldots + (F(x_n) - F(x_{n-1})) =$$

$$= \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) = \sigma_{P_n}$$

$$\implies F(b) - F(a) = \sigma_{P_n}$$

Taking the limits of both sides: $\lim_{n\to\infty} (F(b) - F(a)) = \lim_{n\to\infty} \sigma_{P_n}$

The left-hand side is independent of n and since f is integrable then the limit of the right-hand side is the integral of f, so

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

III. True or false? (16 x 3 points)

1. $\lim_{n\to\infty} a_n = L$ if and only if there exists $\varepsilon > 0$ such that the sequence (a_n) has infinitely many terms closer to L than ε.

False. For example, the sequence $a_n = (-1)^n$ doesn't have a limit, but for $L = 1$ and $\varepsilon = 1$ the sequence has infinitely many terms in the interval $(L - \varepsilon, L + \varepsilon) = (0, 2)$, that is, these terms are closer to $L = 1$ than $\varepsilon = 1$.

2. The sequence $(a_n \cdot b_n)$ is convergent if and only if both (a_n) and (b_n) are convergent.

False. For example, if $a_n = \frac{1}{n}$ and $b_n = n$, then $a_n \cdot b_n = 1$, which is convergent, (a_n) is convergent, but (b_n) is divergent.

3. The sequence $a_n = \cos n$ has a convergent subsequence.

True. Since the sequence is bounded, then by the Bolzano-Weierstrass theorem it has a convergent subsequence.

4. If $a_n \longrightarrow 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

False. For example, $a_n = \frac{1}{n} \longrightarrow 0$, but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

5. If $a_n > 0$ for all $n \in \mathbb{N}$ and the sequence (a_n) is monotonically decreasing, then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent.

False. If the sequence does not tend to zero, then the statement is not true.
For example, if $a_n = 1 + \frac{1}{n} > 0$, then (a_n) is monotonically decreasing, but since $\lim_{n\to\infty} a_n = 1$, then $\lim_{n\to\infty} (-1)^n a_n$ doesn't exist, so by the nth term test the series $\sum_{n=1}^{\infty} (-1)^n a_n$ diverges.

6. If the set $H \subset \mathbb{R}$ contains all of its limit points, then it is closed.

True. We learned the following theorem:
A set $A \subset \mathbb{R}$ is closed if and only if it contains all of its limit points.

7. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and assume that for all sequences (x_n) with non-zero terms for which $\lim_{n\to\infty} x_n = 0$, we have $\lim_{n\to\infty} f(x_n) = A$. Then $\lim_{x\to 0} f(x) = A$.

True. This follows from the sequential criterion for continuity.

8. The function $f(x) = \frac{1}{\ln(x^2)}$ has a jump discontinuity at $x = 0$.

False. The function is even and is defined in a neighbourhood of 0,

$$\lim_{x\to 0\pm 0} x^2 = 0 \implies \lim_{x\to 0\pm 0} \ln(x^2) = -\infty \implies \lim_{x\to 0\pm 0} \frac{1}{\ln(x^2)} = \frac{1}{-\infty} = 0.$$

Since f has a finite limit but it is not defined at $x = 0$, then f has a removable discontinuity at this point.

9. The equation $x^5 = 10\,x^2 - 3$ has a real solution in the interval $[0, 1]$.

True. Solving the equation for $0 \leq x \leq 1$ is equivalent to finding a root of $f(x) = x^5 - 10\,x^2 + 3$ on $[0, 1]$. Then f is continuous, $f(0) = 3 > 0$, $f(1) = -6 < 0$, therefore, by the intermediate value theorem f has a real root on $[0, 1]$. So the equation also has a real solution on $[0, 1]$.

10. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at x_0, then the limit $\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists and is finite.

False. If the above limit exists and is finite then it means that f is differentiable at x_0 and the value of the derivative if $f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$. We learned that differentiability at a point implies continuity, however, the converse is not true. For example, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = |x|$ is continuous but not differentiable at $x_0 = 0$.

11. The function $f(x) = e^x + \arctan(x)$ is invertible on $\mathbb{R}$.

True. Since $f'(x) = e^x + \frac{1}{1 + x^2} > 0$ for all $x \in \mathbb{R}$, then then f is strictly monotonically increasing on $\mathbb{R}$, so f is invertible on $\mathbb{R}$.

12. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$. Then f has both a minimum and a maximum on $[a, b]$ only if f is differentiable on (a, b).

False. Assuming continuity of f on a closed interval is sufficient for the existence of a minimum and a maximum. For example, if $f(x) = |x|$ for $-1 \leq x \leq 1$, then f has a minimum at $x = 0$, but f is not differentiable at $x = 0$.

13. Assume that f is at least two times differentiable on $\mathbb{R}$. If $f''(x_0) = 0$, then f has an inflection point at x_0.

False. For example, if $f(x) = x^4$, then $f''(x) = 12\,x^2$, so $f''(0) = 0$, but f doesn’t have an inflection point at $x_0 = 0$.

14. The partial fraction decomposition of $f(x) = \frac{x + 6}{(x + 1)(x^4 - 1)}$ contains the term $\frac{A\,x + B}{x^2 + 1}$.

True. The partial fraction decomposition of $f(x) = \frac{x + 6}{(x + 1)(x^2 - 1)(x^2 + 1)} = \frac{x + 6}{(x + 1)^2 (x - 1)(x^2 + 1)}$ is

$$f(x) = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} + \frac{E\,x + F}{x^2 + 1}.$$

15. The function $f(x) = \text{sgn}(x)$ has an antiderivative on $[-1, 1]$.

False. Since f has a jump discontinuity, then there is no such function $F : [-1, 1] \longrightarrow \mathbb{R}$ for which
$F'(x) = f(x)$, that is, f doesn't have an antiderivative on $[-1, 1]$.
By Darboux's theorem if $-1 < y < 1$ then $F'(x) = y$ should hold for some $x \in [-1, 1]$,
but if $y = \frac{1}{2}$, then there is no such x.

16. If $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and has a finite number of discontinuities, then f is Riemann-integrable on $[a, b]$.

True. This is a theorem that we learned. It is also a consequence of Lebesgue's theorem.