Calculus 1, Midterm Test 1

28th October, 2021

Name: ___________________________________ **Neptun code:** __________

1.	2.	3.	4.	5.	6.	7.	8.	9.	$\sum$

1. (10 points) Solve the following equation on the set of complex numbers:

$z^2 = z + 3\bar{z}$

2. (9 points) Let $a_n = \dfrac{6n^3 + n^2 - 7}{3n^3 - 2n + 1}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.01$.

3. (9+9 points) Find the limit of the following sequences:

a) $a_n = \sqrt[n]{\dfrac{3^n \cdot n + 5^n}{n^2 + 2}}$ b) $a_n = \left(\dfrac{n^2 + 4}{n^2 + 2}\right)^{n^3}$

4. (12 points) Let $a_1 = 2$ and $a_{n+1} = \sqrt{a_n - 2} + 4$ for all $n \in \mathbb{N}$. Prove that (a_n) is convergent and calculate its limit.

5. (9 points) Find the liminf and limsup of $a_n = \sqrt{n^2 + 3n} + (-1)^n \cdot \sqrt{n^2 + 5n + 3}$.

6. (6 points) Calculate the sum of the following series: $\displaystyle\sum_{n=1}^{\infty} \frac{3 \cdot 2^n + (-2)^n \cdot 3^{-n}}{6^n}$

7. (9+9+9 points) Decide whether the following series are convergent or divergent:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$ b) $\displaystyle\sum_{n=1}^{\infty} 10^n \left(\frac{2n+1}{2n+5}\right)^{n^2}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 \cdot \ln n + \sqrt{n+1}}{n^3 - n + 3}$

8. (9 points) For what values of $x \in \mathbb{R}$ does the following series converge?

$\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^2 \cdot 3^n} x^n$

9.* (10 points - BONUS): Calculate the limit of the following sequence:

$$a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+3n}}$$

Solutions

1. (10 points) Solve the following equation on the set of complex numbers:

$z^2 = z + 3\bar{z}$

Solution. Let $z = x + y\,i$ $(x, y \in \mathbb{R})$. Then $z^2 = (x^2 - y^2) + 2\,x\,y\,i$ and $\bar{z} = x - y\,i$ **(2p)**.

We obtain the following equation system:

(1) $x^2 - y^2 = 4\,x$

(2) $2\,x\,y = -2\,y$ **(2p)**

From the second equation $2\,y(x+1) = 0 \implies y = 0$ or $x = -1$ **(2p)**

If $y = 0$ then from the first equation $x = 0$ or $x = 4$ **(1p)**

If $x = -1$ then from the first equation $y = \pm\sqrt{5}$ **(1p)**

The solutions are $z_1 = 0$, $z_2 = 4$, $z_3 = -1 + i\sqrt{5}$, $z_4 = -1 - i\sqrt{5}$ **(2p)**

2. (9 points) Let $a_n = \dfrac{6\,n^3 + n^2 - 7}{3\,n^3 - 2\,n + 1}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.01$.

Solution. $a_n = \dfrac{6\,n^3 + n^2 - 7}{3\,n^3 - 2\,n + 1} = \dfrac{6 + \frac{1}{n} - \frac{7}{n^3}}{3 - \frac{2}{n^2} + \frac{1}{n^3}} \longrightarrow \dfrac{6 + 0 - 0}{3 - 0 + 0} = 2$ **(1p)**

Let $\varepsilon > 0$. We have to find $N(\varepsilon) \in \mathbb{N}$ such that if $n > N$ then $|\,a_n - A\,| < \varepsilon$. $(A = 2)$ **(1p)**

$$|\,a_n - A\,| = \left|\frac{6\,n^3 + n^2 - 7}{3\,n^3 - 2\,n + 1} - 2\right| = \left|\frac{6\,n^3 + n^2 - 7 - 2\cdot(3\,n^3 - 2\,n + 1)}{3\,n^3 - 2\,n + 1}\right| =$$

$$= \left|\frac{n^2 + 4\,n - 9}{3\,n^3 - 2\,n + 1}\right| \overset{\text{if } n \geq 2}{=} \frac{n^2 + 4\,n - 9}{3\,n^3 - 2\,n + 1} \;\textbf{(2 p)}\; \leq \frac{n^2 + 4\,n^2 + 0}{3\,n^3 - 2\,n^3 + 0} = \frac{5}{n} < \varepsilon \iff n > \frac{5}{\varepsilon},\; \textbf{(2p)}$$

so with the choice $N(\varepsilon) \geq \max\left\{2, \left[\frac{5}{\varepsilon}\right]\right\}$ the definition holds. **(2p)**

If $\varepsilon = 0.01$ then $N \geq 500$. **(1p)**

3. (9+9 points) Find the limit of the following sequences:

a) $a_n = \sqrt[n]{\dfrac{3^n \cdot n + 5^n}{n^2 + 2}}$ b) $a_n = \left(\dfrac{n^2 + 4}{n^2 + 2}\right)^{n^3}$

Solution. a) An upper estimation:

$$a_n = \sqrt[n]{\frac{3^n \cdot n + 5^n}{n^2 + 2}} \leq \sqrt[n]{\frac{5^n \cdot n + 5^n \cdot n}{1}} = \sqrt[n]{2 \cdot 5^n \cdot n} = 5 \cdot \sqrt[n]{2} \cdot \sqrt[n]{n} \longrightarrow 5 \cdot 1 \cdot 1 = 5 \;\textbf{(4p)}$$

A lower estimation:

$$a_n = \sqrt[n]{\frac{3^n \cdot n + 5^n}{n^2 + 2}} \geq \sqrt[n]{\frac{0 + 5^n}{n^2 + 2\,n^2}} = \sqrt[n]{\frac{5^n}{3\,n^2}} = \frac{5}{\sqrt[n]{3} \cdot \left(\sqrt[n]{n}\right)^2} \longrightarrow \frac{5}{1 \cdot 1^2} = 5 \;\textbf{(4p)}$$

so by the sandwich theorem, $a_n \longrightarrow 5$. **(1p)**

b) Let $b_n = \left(\frac{n^2+4}{n^2+2}\right)^{n^2}$, then $a_n = (b_n)^n$ and $b_n = \frac{\left(1+\frac{4}{n^2}\right)^{n^2}}{\left(1+\frac{2}{n^2}\right)^{n^2}} \longrightarrow \frac{e^4}{e^2} = e^2$ **(3p)**.

Since $b_n \longrightarrow e^2$ then there exists $N \in \mathbb{N}$ such that if $n > N$ then $b_n > 2$.
So if $n > N$ then $a_n = (b_n)^n > 2^n$. Since $2^n \longrightarrow \infty$, then $a_n \longrightarrow \infty$. **(6p)**

4. (12 points) Let $a_1 = 2$ and $a_{n+1} = \sqrt{a_n - 2} + 4$ for all $n \in \mathbb{N}$. Prove that (a_n) is convergent and calculate its limit.

Solution. If $\exists \lim\limits_{n\to\infty} a_n = A$, then $A = \sqrt{A-2} + 4 \implies (A-4)^2 = A - 2$
$\implies A^2 - 9A + 18 = (A-3)(A-6) = 0 \implies A_1 = 3,\ A_2 = 6$ **(3p)**
It can be verified that $A = 3$ is not a solution, so if the limit exists then $A = 6$. **(1p)**
Monotonicity:

(1) $a_1 = 2 < a_2 = \sqrt{2-2} + 4 = 4$
(2) Assume that $2 < a_n < a_{n+1}$
(3) Then $0 < a_n - 2 < a_{n+1} - 2 \implies a_{n+1} = \sqrt{a_n - 2} + 4 < \sqrt{a_{n+1} - 2} + 4 = a_{n+2}$.
So (a_n) is monotonically increasing. **(3p)**

Boundedness:

(1) $a_1 = 2 < 6$
(2) Assume that $2 < a_n < 6$
(3) Then $0 < a_n - 2 < 4 \implies a_{n+1} = \sqrt{a_n - 2} + 4 < \sqrt{6-2} + 4 = 6$
So (a_n) is bounded above. **(3p)**

Since (a_n) is monotonically increasing and bounded above then it is convergent and $\lim\limits_{n\to\infty} a_n = 6$. **(2p)**

5. (9 points) Find the liminf and limsup of $a_n = \sqrt{n^2 + 3n} + (-1)^n \cdot \sqrt{n^2 + 5n + 3}$.

Solution. If n is even, then $a_n = \sqrt{n^2+3n} + \sqrt{n^2+5n+3} = \infty + \infty = \infty$ **(2p)**

If n is odd, then $a_n = \left(\sqrt{n^2+3n} - \sqrt{n^2+5n+3}\right) \cdot \frac{\sqrt{n^2+3n} + \sqrt{n^2+5n+3}}{\sqrt{n^2+3n} + \sqrt{n^2+5n+3}} =$ **(1p)**

$$= \frac{(n^2+3n) - (n^2+5n+3)}{\sqrt{n^2+3n} + \sqrt{n^2+5n+3}} = \frac{-2n-3}{\sqrt{n^2+3n} + \sqrt{n^2+5n+3}} =$$

$$= \frac{n}{\sqrt{n^2}} \cdot \frac{-2-\frac{3}{n}}{\sqrt{1+\frac{3}{n}} + \sqrt{1+\frac{5}{n}+\frac{3}{n^2}}} \longrightarrow \frac{-2-0}{\sqrt{1+0} + \sqrt{1+0+0}} = -1 \text{ (4p)}$$

$\implies \limsup a_n = \infty,\ \liminf a_n = -1$ **(2p)**

6. (6 points) Calculate the sum of the following series: $\sum_{n=1}^{\infty} \frac{3 \cdot 2^n + (-2)^n \cdot 3^{-n}}{6^n}$

Solution. $\sum_{n=1}^{\infty} \frac{3 \cdot 2^n + (-2)^n \cdot 3^{-n}}{6^n} = \sum_{n=1}^{\infty} \left(3 \cdot \left(\frac{2}{6}\right)^n + \left(\frac{-2}{6 \cdot 3}\right)^n\right) = \sum_{n=1}^{\infty} \left(3 \cdot \left(\frac{1}{3}\right)^n + \left(-\frac{1}{9}\right)^n\right) =$ **(2p)**

$= 3 \cdot \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{-\frac{1}{9}}{1-\left(-\frac{1}{9}\right)}$ **(4p)** $= \frac{3}{2} - \frac{1}{10} = \frac{7}{5}$

7. (9+9+9 points) Decide whether the following series are convergent or divergent:

a) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$ b) $\sum_{n=1}^{\infty} 10^n \left(\frac{2n+1}{2n+5}\right)^{n^2}$ c) $\sum_{n=1}^{\infty} \frac{n^2 \cdot \ln n + \sqrt{n+1}}{n^3 - n + 3}$

Solution. a) Let $a_n = \frac{(2n)!}{n^{2n}}$. By the ratio test:

$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{(2n)!}$ **(3 *p*)** $= \frac{(2n+2)\cdot(2n+1)}{(n+1)^2} \cdot \frac{n^{2n}}{(n+1)^{2n}} = \frac{2\cdot(2n+1)}{(n+1)} \cdot \left(\frac{n}{n+1}\right)^{2n} =$

$= \frac{4n+2}{n+1} \cdot \left(\frac{1}{1+\frac{1}{n}}\right)^{2n} = \frac{4+\frac{2}{n}}{1+\frac{1}{n}} \cdot \frac{1}{\left(\left(1+\frac{1}{n}\right)^n\right)^2}$ **(3 *p*)** $\longrightarrow 4 \cdot \frac{1}{e^2}$ **(2p)** $< 1 \Longrightarrow$ the series $\sum_{n=1}^{\infty} a_n$ is convergent **(1p)**

b) Let $b_n = 10^n \left(\frac{2n+1}{2n+5}\right)^{n^2}$. By the root test:

$\sqrt[n]{b_n} = 10 \cdot \left(\frac{2n+1}{2n+5}\right)^n$ **(3p)** $= 10 \cdot \frac{\left(1+\frac{1}{2n}\right)^n}{\left(1+\frac{5}{2n}\right)^n}$ **(3p)** $\longrightarrow 10 \cdot \frac{e^{\frac{1}{2}}}{e^{\frac{5}{2}}} = \frac{10}{e^2}$ **(2p)** $> 1 \Longrightarrow$ the series $\sum_{n=1}^{\infty} b_n$ is divergent **(1p)**

c) $c_n = \frac{n^2 \cdot \ln n + \sqrt{n+1}}{n^3 - n + 3} \geq \frac{n^2 \cdot 1 + 0}{n^3 + 0 + 3n^3} = \frac{1}{4n} > 0$ **(6p)** and $\sum_{n=1}^{\infty} \frac{1}{4n}$ is divergent, so by the comparison test, the series $\sum_{n=1}^{\infty} c_n$ is divergent. **(3p)**

8. (9 points) For what values of $x \in \mathbb{R}$ does the following series converge?

$\sum_{n=1}^{\infty} \frac{n+2}{n^2 \cdot 3^n} x^n$

Solution. The coefficients are $a_n = \frac{n+2}{n^2 \cdot 3^n}$ and the center is $x_0 = 0$.

$\sqrt[n]{a_n} = \sqrt[n]{\frac{n+2}{n^2 \cdot 3^n}} = \frac{\sqrt[n]{n+2}}{\left(\sqrt[n]{n}\right)^2 \cdot 3} \longrightarrow \frac{1}{1^2 \cdot 3} = \frac{1}{3} = \frac{1}{R} \Longrightarrow R = 3.$ **(3p)**

$\sqrt[n]{n+2} \longrightarrow 1$ by the sandwich theorem, since $1 \leq \sqrt[n]{n+2} \leq \sqrt[n]{n+2n} = \sqrt[n]{3} \cdot \sqrt[n]{n} \longrightarrow 1 \cdot 1 = 1$.

Let H denote the domain of convergence. Then $(-3, 3) \subset H \subset [-3, 3]$.

The endpoints of H:

If $x = x_0 + R = 3$ then the series is $\sum_{n=1}^{\infty} \frac{n+2}{n^2 \cdot 3^n} 3^n = \sum_{n=1}^{\infty} \frac{n+2}{n^2}$. Since $\frac{n+2}{n^2} \geq \frac{n+0}{n^2} = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then by the comparison test, $\sum_{n=1}^{\infty} \frac{n+2}{n^2}$ also diverges. $\Longrightarrow 3 \notin H$. **(2p)**

If $x = x_0 - R = -3$ then the series is $\sum_{n=1}^{\infty} \frac{n+2}{n^2 \cdot 3^n} (-3)^n = \sum_{n=1}^{\infty} (-1)^n \frac{n+2}{n^2}$. This is a Leibniz series (or: the sum of two Leibniz series), so it is convergent. $\Longrightarrow -3 \in H$. **(2p)**

The domain of convergence is $H = [-3, 3)$. **(2p)**

9.* (10 points - BONUS): Calculate the limit of the following sequence:

$$a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+3n}}$$

Solution. An upper estimation:

$$a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+3n}} \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \ldots + \frac{1}{\sqrt{n^2+1}} =$$

$$= \frac{3n}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2}} \cdot \frac{3}{\sqrt{1+\frac{1}{n^2}}} \longrightarrow \frac{3}{\sqrt{1+0}} = 3 \quad \textbf{(4p)}$$

A lower estimation:

$$a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+3n}} \geq \frac{1}{\sqrt{n^2+3n}} + \frac{1}{\sqrt{n^2+3n}} + \ldots + \frac{1}{\sqrt{n^2+3n}} =$$

$$= \frac{3n}{\sqrt{n^2+3n}} = \frac{n}{\sqrt{n^2}} \cdot \frac{3}{\sqrt{1+\frac{3}{n}}} \longrightarrow \frac{3}{\sqrt{1+0}} = 3 \ \textbf{(4p)}$$

so by the sandwich theorem, $a_n \longrightarrow 3$. **(2p)**