Calculus 1, Midterm test 1

26th October, 2023

Name: ____________________________________ Neptun code: __________

1.	2.	3.	4.	5.	6.	7.	8.	9.	$\sum$

1. (9 points) Let $a_n = \dfrac{4n^2+3n-1}{2n^2-n+17}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.01$.

2. (9 points) Find the limit of the following sequence: $a_n = \dfrac{1}{3n+1-\sqrt{9n^2+5n}}$

3. (9+9 points) Find the limit of the following sequences:

a) $a_n = \left(\dfrac{3n^2+4}{3n^2+1}\right)^n$ b) $b_n = \left(\dfrac{3n+2}{4n+3}\right)^{n+3}$

4. (12 points) Let $a_1 = 2$ and $a_{n+1} = 5 - \dfrac{4}{a_n}$ for all $n \in \mathbb{N}$.

(Then $a_2 = 3$, $a_3 \approx 3.67, \ldots$). Prove that (a_n) is convergent and calculate its limit.

5. (9 points) Find the liminf and limsup of $a_n = (-1)^n \cdot \sqrt[n]{\dfrac{10n^3-n}{n^4+5}}$.

6. (6 points) Calculate the sum of the following series: $\displaystyle\sum_{n=2}^{\infty} \frac{10+(-6)^n}{3\cdot 2^{3n+1}}$.

7. (9+9+9 points) Decide whether the following series are convergent or divergent:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+2n-7}{4n^3\cdot\sqrt{n}+9n^2-n+1}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2+n)^n}{(n+1)!}$ c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+6}{n^2+4}\right)^{n^3} \cdot \frac{n^2}{9^{n+1}}$

8. (10 points) Find the interval of convergence of the following power series: $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2\cdot 4^n}\cdot(x+3)^n$

9.* (10 points - BONUS):

Let (b_n) be the following periodic sequence: 3, 4, 5, 3, 4, 5, 3, 4, 5, … .

Let (c_n) be the following sequence: $c_n = \dfrac{\left(b_n - 3 + \frac{1}{n}\right)^n}{2^n}$.

Find the accumulations points and the liminf and limsup of the sequences (b_n) and (c_n).

Solutions

1. (9 points) Let $a_n = \dfrac{4n^2+3n-1}{2n^2-n+17}$. Find the limit of a_n and provide a threshold index N for $\varepsilon = 0.01$.

Solution. $a_n = \dfrac{4n^2+3n-1}{2n^2-n+17} = \dfrac{4+\frac{3}{n}-\frac{1}{n^2}}{2-\frac{1}{n}+\frac{17}{n^2}} \longrightarrow \dfrac{4+0-0}{2-0+0} = 2$ **(1p)**

Let $\varepsilon > 0$. We have to find $N(\varepsilon) \in \mathbb{N}$ such that if $n > N$ then $|a_n - A| < \varepsilon$. $(A = 2)$ **(1p)**

$$|a_n - A| = \left|\frac{4n^2+3n-1}{2n^2-n+17} - 2\right| = \left|\frac{4n^2+3n-1-2\cdot(2n^2-n+17)}{2n^2-n+17}\right| = \left|\frac{5n-35}{2n^2-n+17}\right| \overset{\text{if } n\geq 7}{=} \frac{5n-35}{2n^2-n+17}$$ **(2p)**

$$\frac{5n-35}{2n^2-n+17} \leq \frac{5n+0}{2n^2-n^2+0} = \frac{5n}{n^2} = \frac{5}{n} < \varepsilon \iff n > \frac{5}{\varepsilon}$$ **(3p)**

so with the choice $N(\varepsilon) \geq \max\left\{7, \left[\frac{5}{\varepsilon}\right]\right\}$ the definition holds. **(1p)**

If $\varepsilon = 0.01$ then $N \geq \left[\frac{5}{0.01}\right] = 500$. **(1p)**

2. (9 points) Find the limit of the following sequence: $a_n = \dfrac{1}{3n+1-\sqrt{9n^2+5n}}$

Solution. $a_n = \dfrac{1}{3n+1-\sqrt{9n^2+5n}} \cdot \dfrac{3n+1+\sqrt{9n^2+5n}}{3n+1+\sqrt{9n^2+5n}} =$ **(2p)**

$$= \frac{3n+1+\sqrt{9n^2+5n}}{(3n+1)^2-(9n^2+5n)} = \frac{3n+1+\sqrt{9n^2+5n}}{9n^2+6n+1-(9n^2+5n)} = \frac{3n+1+\sqrt{9n^2+5n}}{n+1}$$

$= \dfrac{n}{n} \cdot \dfrac{3+\frac{1}{n}+\sqrt{9+\frac{5}{n}}}{1+\frac{1}{n}}$ **(5p)** $\longrightarrow \dfrac{3+0+\sqrt{9+0}}{1+0} = 6$ **(2p)**

3. (9+9 points) Find the limit of the following sequences:

a) $a_n = \left(\dfrac{3n^2+4}{3n^2+1}\right)^n$ b) $b_n = \left(\dfrac{3n+2}{4n+3}\right)^{n+3}$

Solution. a) $a_n^n = \left(\dfrac{3n^2+4}{3n^2+1}\right)^{n^2} = \dfrac{\left(1+\frac{4}{3n^2}\right)^{n^2}}{\left(1+\frac{1}{3n^2}\right)^{n^2}} \longrightarrow \dfrac{e^{\frac{4}{3}}}{e^{\frac{1}{3}}} = e.$ **(4p)**

Since $2 < e < 3$ then $2 < a_n^n < 3$ if n is large enough. **(2p)**

Then $\sqrt[n]{2} < a_n < \sqrt[n]{3}$, and since $\sqrt[n]{2} \longrightarrow 1$ and $\sqrt[n]{3} \longrightarrow 1$,

then by the sandwich theorem $a_n \longrightarrow 1$. **(3p)**

b) $b_n = \left(\frac{3n+2}{4n+3}\right)^{n+3} = \frac{\left(3n\left(1+\frac{2}{3n}\right)\right)^{n+3}}{\left(4n\left(1+\frac{3}{4n}\right)\right)^{n+3}} = \left(\frac{3}{4}\right)^{n+3} \cdot \frac{\left(1+\frac{2}{3n}\right)^n}{\left(1+\frac{3}{4n}\right)^n} \cdot \frac{\left(1+\frac{2}{3n}\right)^3}{\left(1+\frac{3}{4n}\right)^3}$ **(6p)** $\longrightarrow 0 \cdot \frac{e^{\frac{2}{3}}}{e^{\frac{3}{4}}} \cdot \frac{1}{1} = 0$ **(3p)**

4. (12 points) Let $a_1 = 2$ and $a_{n+1} = 5 - \frac{4}{a_n}$ for all $n \in \mathbb{N}$.

(Then $a_2 = 3$, $a_3 \approx 3.67, \ldots$). Prove that (a_n) is convergent and calculate its limit.

Solution. If $\exists \lim\limits_{n\to\infty} a_n = A$ then $A = 5 - \frac{4}{A} \iff A^2 - 5A + 4 = (A-1)(A-4) = 0$

$\iff A_1 = 1,\ A_2 = 4$ **(3p)**.

Boundedness: we prove by induction that $1 < a_n < 4$ for all $n \in \mathbb{N}$.

(1) $1 < a_1 = 2 < 4$

(2) Assume that $1 < a_n < 4$

(3) Then $1 > \frac{1}{a_n} > \frac{1}{4} \implies -4 < -\frac{4}{a_n} < -1 \implies 1 < a_{n+1} = 5 - \frac{4}{a_n} < 4$

So (a_n) is bounded above. **(3p)**

Monotonicity: we prove by induction that (a_n) is monotonically increasing, that is, $a_n < a_{n+1}\ \forall n \in \mathbb{N}$.

(1) $a_1 = 2 < a_2 = 3$

(2) Assume that $a_n < a_{n+1}$

(3) Then $\frac{1}{a_n} > \frac{1}{a_{n+1}}$ (since $a_n > 1 > 0$) $\implies \frac{-4}{a_n} < \frac{-4}{a_{n+1}} \implies a_{n+1} = 5 - \frac{4}{a_n} < 5 - \frac{4}{a_{n+1}} = a_{n+2}$

So (a_n) is monotonically increasing. **(3p)**

Since (a_n) is monotonically increasing and bounded above then it is convergent.
Since $a_1 = 2$ and the sequence is monotonically increasing then $A = 1$ cannot be the limit.
So $\lim\limits_{n\to\infty} a_n = 4$. **(3p)**

5. (9 points) Find the liminf and limsup of $a_n = (-1)^n \cdot \sqrt[n]{\frac{10n^3 - n}{n^4 + 5}}$.

Solution. Let $b_n = \sqrt[n]{\frac{10n^3 - n}{n^4 + 5}}$. An upper estima-

tion:

$$b_n = \sqrt[n]{\frac{10n^3 - n}{n^4 + 5}} \le \sqrt[n]{\frac{10n^3 + 0}{n^4 + 0}} = \sqrt[n]{\frac{10}{n}} = \sqrt[n]{10} \cdot \frac{1}{\sqrt[n]{n}} \longrightarrow 1 \cdot \frac{1}{1} = 1 \text{ (3p)}$$

A lower estimation:

$$a_n = \sqrt[n]{\frac{10n^3 - n}{n^4 + 5}} \ge \sqrt[n]{\frac{10n^3 - n^3}{n^4 + 5n^4}} = \sqrt[n]{\frac{9}{6n}} = \sqrt[n]{\frac{9}{6}} \cdot \frac{1}{\sqrt[n]{n}} \longrightarrow 1 \cdot \frac{1}{1} = 1 \text{ (3p)}$$

so by the sandwich theorem, $b_n \longrightarrow 1$. **(1p)**

If n is even, then $a_n = b_n \longrightarrow 1$ and if n is odd then $a_n = -b_n \longrightarrow -1$, so
$\liminf a_n = -1$ and $\limsup a_n = 1$. **(2p)**

6. (6 points) Calculate the sum of the following series: $\sum_{n=2}^{\infty} \frac{10+(-6)^n}{3\cdot 2^{3n+1}}$.

Solution. $\sum_{n=2}^{\infty} \frac{10+(-6)^n}{3\cdot 2^{3n+1}} = \sum_{n=2}^{\infty} \frac{10+(-6)^n}{6\cdot 8^n} = \sum_{n=2}^{\infty}\left(\frac{10}{6}\cdot\left(\frac{1}{8}\right)^n + \frac{1}{6}\cdot\left(\frac{-6}{8}\right)^n\right) =$ **(2p)**

$= \frac{10}{6}\,\frac{\left(\frac{1}{8}\right)^2}{1-\frac{1}{8}} + \frac{1}{6}\,\frac{\left(-\frac{3}{4}\right)^2}{1-\left(-\frac{3}{4}\right)}$ **(4p)** $= \frac{5}{168} + \frac{3}{56} = \frac{1}{12}$

7. (9+9+9 points) Decide whether the following series are convergent or divergent:

a) $\sum_{n=1}^{\infty} \frac{3n^2+2n-7}{4n^3\cdot\sqrt{n}+9n^2-n+1}$ b) $\sum_{n=1}^{\infty} \frac{(2+n)^n}{(n+1)!}$ c) $\sum_{n=1}^{\infty} \left(\frac{n^2+6}{n^2+4}\right)^{n^3}\cdot\frac{n^2}{9^{n+1}}$

Solution. a) Let $a_n = \frac{3n^2+2n-7}{4n^3\cdot\sqrt{n}+9n^2-n+1}$. Then for large enough n we have

$0 < a_n \le \frac{3n^2+2n^2+0}{4n^3\cdot\sqrt{n}+0-n^3\cdot\sqrt{n}+0} = \frac{5n^2}{3n^3\cdot\sqrt{n}} = \frac{5}{3}\cdot\frac{1}{n^{3/2}}$. **(6p)**

Since $\sum_{n=1}^{\infty} \frac{5}{3}\cdot\frac{1}{n^{3/2}}$ is convergent (p-series with $p = \frac{3}{2} > 0$) then by the comparison test

$\sum_{n=1}^{\infty} a_n$ is also convergent. **(3p)**

b) Let $a_n = \frac{(2+n)^n}{(n+1)!}$. By the ratio test:

$\frac{a_{n+1}}{a_n} = \frac{(3+n)^{n+1}}{(n+2)!}\cdot\frac{(n+1)!}{(2+n)^n}$ **(3p)** $= \frac{(3+n)^{n+1}}{(n+2)}\cdot\frac{1}{(2+n)^n} = \frac{(n+3)^{n+1}}{(n+2)^{n+1}} = \frac{\left(1+\frac{3}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}\cdot\frac{1+\frac{3}{n}}{1+\frac{2}{n}} \longrightarrow \frac{e^3}{e^2}\cdot\frac{1}{1} = e > 1$ **(5p)**

$\Longrightarrow$ the series $\sum_{n=1}^{\infty} a_n$ is divergent **(1p)**

c) Let $a_n = \left(\frac{n^2+6}{n^2+4}\right)^{n^3}\cdot\frac{n^2}{9^{n+1}}$. By the root test:

$\sqrt[n]{a_n} = \left(\frac{n^2+6}{n^2+4}\right)^{n^2}\cdot\frac{\left(\sqrt[n]{n}\right)^2}{\sqrt[n]{9}\cdot 9}$ **(3p)** $= \frac{\left(1+\frac{6}{n^2}\right)^{n^2}}{\left(1+\frac{4}{n^2}\right)^{n^2}}\cdot\frac{\left(\sqrt[n]{n}\right)^2}{\sqrt[n]{9}\cdot 9} \longrightarrow \frac{e^6}{e^4}\cdot\frac{1^2}{1\cdot 9} = \frac{e^2}{9} < 1$ **(5p)**

$\Longrightarrow$ the series $\sum_{n=1}^{\infty} a_n$ is convergent **(1p)**

8. (10 points) Find the interval of convergence of the following power series: $\sum_{n=1}^{\infty} \frac{2n+1}{n^2\cdot 4^n}\cdot(x+3)^n$

Solution. The coefficients are $a_n = \frac{2n+1}{n^2\cdot 4^n}$ and the center is $x_0 = -3$.

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{2n+1}{n^2 \cdot 4^n}} = \frac{\sqrt[n]{2n+1}}{\left(\sqrt[n]{n}\right)^2 \cdot 4} \longrightarrow \frac{1}{1^2 \cdot 4} = \frac{1}{4} = \frac{1}{R} \implies R = 4$$

Here we used that $\sqrt[n]{2n+1} \longrightarrow 1$ by the sandwich theorem, since
$1 \leq \sqrt[n]{2n+1} \leq \sqrt[n]{2n+n} = \sqrt[n]{3} \cdot \sqrt[n]{n} \longrightarrow 1 \cdot 1 = 1.$

Let H denote the domain of convergence. The endpoints of H:

If $x = x_0 - R = -3 - 4 = -7$ then the series is $\sum_{n=1}^{\infty} \frac{2n+1}{n^2 \cdot 4^n} \cdot (-4)^n = \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n^2}$.

This is a Leibniz series (or, the sum of two Leibniz series), so it is convergent
$\implies -7 \in H$.

If $x = x_0 + R = -3 + 4 = 1$ then the series is $\sum_{n=1}^{\infty} \frac{2n+1}{n^2 \cdot 4^n} \cdot 4^n = \sum_{n=1}^{\infty} \frac{2n+1}{n^2}$.

Since $\frac{2n+1}{n^2} \geq \frac{n+0}{n^2} = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then by the comparison test,
$\sum_{n=1}^{\infty} \frac{2n+1}{n^2}$ also diverges. $\implies 1 \notin H$.

The domain of convergence is $H = [-7, 1)$

9.* (10 points - BONUS):

Let (b_n) be the following periodic sequence: 3, 4, 5, 3, 4, 5, 3, 4, 5, ...

Let (c_n) be the following sequence: $c_n = \frac{\left(b_n - 3 + \frac{1}{n}\right)^n}{2^n}$.

Find the accumulations points and the liminf and limsup of the sequences (b_n) and (c_n).

Solution. $(b_1, b_2, b_3, \ldots) = (3, 4, 5, 3, 4, 5, \ldots)$

Since (b_n) is constructed from finitely many constant sequences then
its accumulation points are 3, 4, 5 and thus $\liminf b_n = 3$, $\limsup b_n = 5$. **(2p)**

If $n = 3k+1$ $(k \in \mathbb{N}^+)$ then $b_n = 3$, so $c_n = \frac{\left(3 - 3 + \frac{1}{n}\right)^n}{2^n} = \frac{1}{2^n \cdot n^n} \longrightarrow 0$. **(2p)**

If $n = 3k+2$ $(k \in \mathbb{N}^+)$ then $b_n = 4$, so $c_n = \frac{\left(4 - 3 + \frac{1}{n}\right)^n}{2^n} = \frac{1}{2^n} \cdot \left(1 + \frac{1}{n}\right)^n \longrightarrow 0 \cdot e = 0$. **(2p)**

If $n = 3k$ $(k \in \mathbb{N}^+)$ then $b_n = 5$, so

$$c_n = \frac{\left(5 - 3 + \frac{1}{n}\right)^n}{2^n} = \frac{\left(2 + \frac{1}{n}\right)^n}{2^n} = \frac{\left(2\left(1 + \frac{1}{2n}\right)\right)^n}{2^n} = \frac{2^n}{2^n} \cdot \left(1 + \frac{1}{2n}\right)^n = \left(1 + \frac{1}{2n}\right)^n \longrightarrow e^{\frac{1}{2}} = \sqrt{e}.$$ **(2p)**

The accumulation points of (c_n) are 0 and $\sqrt{e}$, so $\liminf c_n = 0$ and $\limsup c_n = \sqrt{e}$. **(2p)**