7. Higher Dimensional Poisson Process

The Process

The Poisson process can be defined in higher dimensions, as a model of random points in *space*. Some specific examples of "random points" are

1. Defects in a sheet of material.
2. Raisins in a cake.
3. Stars in the sky.

Our original construction of the Poisson process on $[0, \infty)$, starting with the interarrival times, does not generalize easily, because this construction depends critically on the *order* of the real numbers. However, the alternate construction in the last section, motivated by the analogy with Bernoulli trials, generalizes very naturally.

For $d \in \mathbb{N}_+$, let λ_d denote d-dimensional measure (technically, Lebesgue measure), defined on subsets of $\mathbb{R}^d$:

$$\lambda_d(A) = \int_A 1 d\boldsymbol{x}$$

Thus, if $d = 2$, $\lambda_2(A)$ is the *area* of $A \subseteq \mathbb{R}^2$ and if $d = 3$, $\lambda_3(A)$ is the *volume* of $A \subseteq \mathbb{R}^3$. Now let $D \subseteq \mathbb{R}^d$ and consider a random process that produces random points in D. For $A \subseteq D$, let $N(A)$ denote the number of random points in A. This collection of random variables $\{N(A) : A \subseteq D\}$ is a **Poisson process** on D with **density parameter** $r > 0$ if the following axioms are satisfied:

1. $N(A)$ has the Poisson distribution with parameter $r\ \lambda_d(A)$.
2. If $(A_1, A_2, \ldots)$ is a sequence of pairwise disjoint subsets of D then $(N(A_1), N(A_2), \ldots)$ is a sequence of independent random variables.

By convention, if $\lambda_d(A) = 0$ then $N(A) = 0$ with probability 1, and if $\lambda_d(A) = \infty$ then $N(A) = \infty$ with probability 1. On the other hand, note that if $0 < \lambda_d(A) < \infty$ then $0 < N(A) < \infty$ with probability 1.

1. In the two-dimensional Poisson process, vary the width w and the rate r. Note the location and shape of the density of N. Now with $w = 3$ and $r = 2$, run the simulation 1000 times with an update frequency of 10. Note the apparent convergence of the empirical density to the true density.

2. Using our previous results on moments of the Poisson distribution, show that for $A \subseteq D$,

a. $\mathbb{E}(N(A)) = r\ \lambda_d(A)$
b. $\operatorname{var}(N(A)) = r\ \lambda_d(A)$
c. $\mathbb{E}\left(u^{N(A)}\right) = \exp\left(r\ \lambda_d(A)(u - 1)\right)$.

In particular, r can be interpreted as the expected density of the random points, justifying the name of the parameter

3. In the two-dimensional Poisson process, vary the width w and the density parameter r. Note the size and location of the mean-standard deviation bar of N. Now with $w = 4$ and $r = 2$, run the simulation 1000 times with an update frequency of 10. Note the apparent convergence of the empirical moments to the true moments.

4. Suppose that defects in a sheet of material follow the Poisson model with an average of 1 defect per 2 square meters. Consider a 5 square meter sheet of material.

a. Find the probability that there will be at least 3 defects.
b. Find the mean and standard deviation of the number of defects.

5. Suppose that raisins in a cake follow the Poisson model with an average of 2 raisins per cubic inch. Consider a slab of cake that measures 3 by 4 by 1 inches.

a. Find the probability that there will be at no more than 20 raisins.
b. Find the mean and standard deviation of the number of raisins.

6. Suppose that the occurrence of trees in a forest of a certain type that exceed a certain critical size follows the Poisson model. In a one-half square mile region of the forest there are 40 trees that exceed the specified size.

a. Estimate the density parameter.
b. Using the estimated density parameter, find the probability of finding at least 100 trees that exceed the specified size in a square mile region of the forest

The Nearest Points

Consider the Poisson process in $\mathbb{R}^2$ with density parameter r. For $t > 0$, let $M_t = N(C_t)$ where

$$C_t = \left\{(x,\ y) \in \mathbb{R}^2 : x^2 + y^2 \le t^2\right\}$$

is the circular region of radius t, centered at the origin. Let $Z_0 = 0$ and for $k \in \mathbb{N}_+$ let Z_k denote the distance of the k^{th} closest point to the origin. Note that Z_k is the analogue of the k^{th} arrival time for the Poisson process on $[0,\ \infty)$.

7. Show that M_t has the Poisson distribution with parameter $\pi\, t^2\, r$.

8. Show that $Z_k \le t$ if and only if $M_t \ge k$.

9. Show that $\pi\, {Z_k}^2$ has the gamma distribution with shape parameter k and rate parameter r.

10. Show that Z_k has probability density function

$$g(z) = \frac{2\,(\pi\, r)^k\, z^{2\,k-1}}{(k-1)!}\, e^{-\pi\, r\, z^2}, \quad z \ge 0$$

11. Show that $\pi Z_k{}^2 - \pi Z_{k-1}{}^2$ for $k \in \mathbb{N}_+$ are independent and each has the exponential distribution with rate parameter r.

The Distribution of the Random Points

Again, the Poisson model defines the most random way to distribute points in space, in a certain sense. Specifically, consider the Poisson process on $\mathbb{R}^d$ with parameter r.

12. Suppose that $A \subseteq \mathbb{R}^d$ contains exactly one random point. Show that the position $\boldsymbol{X} = (X_1, X_2, \ldots, X_d)$ of the point is uniformly distributed in A.

More generally, if A contains n points, then the positions of the points are independent and each is uniformly distributed in A.

13. Suppose that defects in a type of material follow the Poisson model. It is known that a square sheet with side length 2 meters contains one defect. Find the probability that the defect is in a circular region of the material with radius $\frac{1}{4}$ meter.

14. Suppose that $A \subseteq \mathbb{R}^d$ and $B \subseteq A$. Show that the conditional distribution of $N(B)$ given $N(A) = n$ is the binomial distribution with trial parameter n and success parameter

$$p = \frac{\lambda_d(B)}{\lambda_d(A)}$$

.

15. More generally, suppose that $A \subseteq \mathbb{R}^d$ and that A is partitioned into k subsets $(B_1, B_2, \ldots, B_k)$. Show that the conditional distribution of $(N(B_1), N(B_2), \ldots, N(B_k))$ given $N(A) = n$ is the multinomial distribution with parameters n and $(p_1, p_2, \ldots, p_k)$, where for each i,

$$p_i = \frac{\lambda_d(B_i)}{\lambda_d(A)}$$

.

16. Suppose that raisins in a cake follow the Poisson model. A 6 cubic inch piece of the cake with 20 raisins is divided into 3 equal parts. Find the probability that each piece has at least 6 raisins.

Splitting

Suppose that $\{N(A) : A \subseteq D\}$ is a Poisson process in $D \subseteq \mathbb{R}^d$ with density parameter $r > 0$. Splitting this Poisson process works just like splitting of the standard Poisson process. Specifically, suppose that the random points are of k different types and that each random point, independently of the others, is type i with probability p_i. Let $N_i(A)$ denote the number of type i points in a region $A \subseteq D$, for $i \in \{1, 2, \ldots, k\}$. Of course, we must have

$$\sum_{i=1}^k p_i = 1, \quad \sum_{i=1}^k N_i(A) = N(A)$$

17. Show that for $A \subseteq D$,

a. $(N_1(A), N_2(A), \ldots, N_k(A))$ is a sequence of independent random variables.
b. $N_i(A)$ has the Poisson distribution with parameter $r\, p_i\, m_d(A)$ for $i \in \{1, 2, \ldots, k\}$

More generally, $\{N_i(A) : A \subseteq D\}$ is a Poisson processes with density parameter $r\, p_i$ for each $i \in \{1, 2, \ldots, k\}$, and these processes are independent.

18. Suppose that defects in a sheet of material follow the Poisson model, with an average of 5 defects per square meter. Each defect, independently of the others is *mild* with probability 0.5, *moderate* with probability 0.3, or *severe* with probability 0.2. Consider a circular piece of the material with radius 1 meter.

a. Give the mean and standard deviation of the number of defects of each type in the piece.
b. Find the probability that there will be at least 2 defects of each type in the piece.

Simulating Higher Dimensional Poisson Processes

We can simulate a Poisson variable using the general quantile method.

19. Suppose that f is probability density function on $\mathbb{N}$. If U is uniformly distributed on $[0, 1]$ (a random number), show that variable N defined below has probability density function f:

$$N = j \text{ if and only if } \sum_{i=0}^{j-1} f(i) < U \le \sum_{i=0}^{j} f(i)$$

Now we can use the result in the previous exercise to simulate a Poisson process in a region $D \subseteq \mathbb{R}^d$. We will illustrate the method with the rectangle $D = [a, b] \times [c, d] \subseteq \mathbb{R}^2$. where $a < b$ and $c < d$. First, we use a random number U to simulate a random variable N that has the Poisson distribution with parameter $r\,(b - a)\,(d - c)$. Next, if $N = n$, let $(U_1, U_2, \ldots, U_n)$ and $(V_1, V_2, \ldots, V_n)$ be sequences of random numbers, and define

$$X_i = a + (b - a)\, U_i, \quad Y_i = c + (d - c)\, V_i, \quad i \in \{1, 2, \ldots, n\}$$

20. Show that the random points of the Poisson process with rate r on D are simulated by (X_i, Y_i) for $i \in \{1, 2, \ldots, n\}$.