1. Random Experiments

Experiments

Probability theory is based on the paradigm of a **random experiment**; that is, an experiment whose outcome cannot be predicted with certainty, before the experiment is run. In **classical** or **frequency-based** probability theory, we also assume that the experiment can be repeated indefinitely under essentially the same conditions. This assumption is important because the classical theory is concerned with the long-term behavior as the experiment is replicated. By contrast, **subjective** or **belief-based** probabiity theory is concerned with measures of belief about what will happen when we run the experiment. In this view, repeatability is a less crucuial assumption. In any event, a complete definition of a random experiment requires a careful definition of precisely what information about the experiment is being recorded, that is, a careful definition of what constitutes an *outcome*.

The term **parameter** refers to a non-random quantity in a model that, once chosen, remains constant. Many probability models of random experiments have one or more parameters that can be adjusted to fit the physical experiment being modeled.

The subjects of probability and statistics have an inverse relationship of sorts. In probability, we start with a completely specified mathematical model of a random experiment. Our goal is perform various computations that help us understand the random experiment, help us predict what will happen when we run the experiment. In statistics, by contrast, we start with an incompletely specified mathematical model (one or more parameters may be unknown, for example). We run the experiment to collect data, and then use the data to draw inferences about the unnown factors in the mathematical model.

Compound Experiments

Suppose that we have n experiments $(E_1, E_2, ..., E_n)$. We can form a new, compound experiment by performing the n experiments in sequence, E_1 first, and then E_2 and so on, independently of one another. The term **independent** means, intuitively, that the outcome of one experiment has no influence over any of the other experiments. We will make the term mathematically precise later.

In particular, suppose that we have a basic experiment. A fixed number (or even an infinite number) of independent replications of the basic experiment is a new, compound experiment. Many experiments turn out to be compound experiments and moreover, as noted above, (classical) probability theory itself is based on the idea of replicating an experiment.

In particular, suppose that we have a simple experiment with two outcomes. Independent replications of this experiment are referred to as Bernoulli trials, named for Jacob Bernoulli. This is one of the simplest, but most important models in probability. More generally, suppose that we have a simple experiment with k possible outcomes. Independent replications of this experiment are referred to as multinomial trials.

Sometimes an experiment occurs in well-defined stages, but in a *dependent* way, in the sense that the outcome of a given stage is influenced by the outcomes of the previous stages.

Sampling Experiments

In most statistical studies, we start with a population of objects of interest. The objects may be people, memory chips, or acres of corn, for example. Usually there are one or more numerical measurements of interest to us--the height and weight of a person, the lifetime of a memory chip, the amount of rain, amount of fertilizer, and yield of an acre of corn.

Although our interest is in the entire population of objects, this set is usually too large and too amorphous to study. Instead, we collect a random sample of objects from the population and record the measurements of interest of for each object in the sample.

There are two basic types of sampling. If we sample **with replacement**, each item is replaced in the population before the next draw; thus, a single object may occur several times in the sample. If we sample **without replacement**, objects are not replaced in the population. The chapter on Finite Sampling Models explores a number of models based on sampling from a finite population.

Sampling with replacement can be thought of as a compound experiment, based on independent replications of the simple experiment of drawing a single object from the population and recording the measurements of interest. Conversely, a compound experiment that consists of n independent replications of a simple experiment can usually be thought of as a sampling experiment. On the other hand, sampling without replacement is an experiment that consists of dependent stages.

Examples and Applications

Probability theory is often illustrated using simple devices from games of chance: coins, dice, card, spinners, urns with balls, and so forth. Examples based on such devices are pedagogically valuable because of their simplicity and conceptual clarity. On the other hand, it would be a terrible shame if you were to think that probability is *only* about gambling and games of chance. Rather, try to see problems involving coins, dice, etc. as metaphors for more complex and realistic problems.

Coins and Dice

1. Consider the **coin experiment** of tossing n (distinct) coins and recording the score (1 for heads or 0 for tails) for each coin.

   a. Identify a parameter of the experiment.
   b. Identify the experiment as a compound experiment that consists of independent replications of a simple experiment.
   c. Identify the experiment as sampling with replacement from a population. Give the population and the sample size.
   d. Identify the experiment as n Bernoulli trials.

2. In the simulation of the coin experiment, set $n = 5$. Run the simulation 100 times and observe the outcomes.

3. Consider the **dice experiment** of throwing n distinct, k-sided dice (with sides numbered 1 to k), and recording the score on each die. When $k = 6$ we have **standard dice**.

a. Identify two parameters of the experiment.
b. Identify the experiment as a compound experiment that consists of independent replications of a simple experiment.
c. Identify the experiment as sampling with replacement from a population. Give the population and the sample size.
d. Identify the experiment as n multinomial trials.

4. In the simulation of the dice experiment, set $n = 5$. Run the simulation 100 times and observe the outcomes.

5. In the **die-coin experiment**, a standard die is thrown and then a coin is tossed the number of times shown on the die. The sequence of coin scores is recorded (1 for heads and 0 for tails). Identify the experiment as a compound experiment that consists of dependent stages.

6. Run the simulation of the die-coin experiment 100 times and observe the outcomes.

7. In the **coin-die experiment**, a coin is tossed. If the coin lands heads, a red die is thrown and if the coin lands tails, a green die is thrown. The coin score (1 for heads and 0 for tails) and the die score are recorded. Identify the experiment as a compound experiment that consists of dependent stages.

8. Run the simulation of the coin-die experiment 100 times and observe the outcomes.

Cards

A standard **card deck** can be modeled by the product set

$$D = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, j, q, k\} \times \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$$

where the first coordinate encodes the **denomination** or **kind** (ace, 2-10, jack, queen, king) and where the second coordinate encodes the **suit** (clubs, diamonds, hearts, spades). Sometimes we represent a card as a *string* rather than an ordered pair (for example q♥).

9. Consider the card experiment that consists of dealing n cards from a standard deck.

a. Identify the experiment as a compound experiment that consists of dependent stages.
b. Identify the experiment as sampling without replacement from a population. Give the population and the sample size.

10. In the simulation of the card experiment, set $n = 5$. Run the simulation 100 times and observe the outcomes.

The special case $n = 5$ is the **poker experiment** and the special case $n = 13$ is the **bridge experiment**.

Urn Models

11. An urn contains m distinct balls, labeled from 1 to m. The experiment consists of selecting n balls from the urn, without replacement, and recording the sequence of ball numbers.

a. Identify two parameters of the experiment.
b. Identify the experiment as a compound experiment that consists of dependent stages.
c. Identify the experiment as sampling without replacement from a population. Give the population and the sample size.

12. Consider the basic urn model of the previous exercise. Suppose that r of the m balls are red and the remaining $m - r$ balls are green. Identify an additional parameter of the model. This experiment is a metaphor for sampling from a general **dichotomous** population

13. In the simulation of the urn experiment, set $m = 100$, $r = 40$, and $n = 25$. Run the experiment 100 times and observe the results.

14. An urn initially contains m balls; r are red and $m - r$ are green. A ball is selected from the urn and removed, and then replaced with k balls of the same color. The process is repeated. This is known as **Pólya's urn model**, named after George Pólya.

a. Identify the parameters of the experiment.
b. Argue that $k = 0$ is equivalent to sampling without replacement.
c. Argue that $k = 1$ is equivalent to sampling with replacement.

Buffon's Coin Experiment

15. Buffon's coin experiment consists of tossing a coin with radius $r \leq \frac{1}{2}$ on a floor covered with square tiles of side length 1. The coordinates of the center of the coin are recorded, relative to axes through the center of the square, parallel to the sides. The experiment is named for Compte de Buffon.

a. Identify a parameter of the experiment
b. Identify the experiment as a compound experiment that consists of independent replications of a simple experiment.
c. Identify the experiment as sampling with replacement. Give the population and sample size.

16. In the simulation of Buffon's coin experiment, set $r = 0.1$. Run the experiment 100 times and observe the outcomes.

Reliability

In the usual model of **structural reliability**, a system consists of n components, each of which is either **working** or **failed**. The states of the components are uncertain, and hence define a random experiment. The system as a whole is also either working or failed, depending on the states of the components and how the components are connected. For example, a **series system** works if and only if each component works, while a **parallel system** works if and only if at least one component works. More generally, a **k out of n system** works if at least k components work.

17. Consider the k out of n reliability model.

a. Identify two parameters.
b. What value of k gives a series system?
c. What value of k gives a parallel system?

The reliability model above is a **static** model. It can be extended to a **dynamic** model by assuming that each component is initially working, but has a random time until failure. The system as a whole would also have a random time until failure that would depend on the component failure times and the structure of the system.

Genetics

In ordinary sexual reproduction, the genetic material of a child is a random combination of the genetic material of the parents. Thus, the birth of a child is a random experiment with respect to outcomes such as eye color, hair type, and many other physical traits. We are often particularly interested in the random transmission of traits and the random transmission of genetic disorders.

For example, let's consider an overly simplified model of an **inherited trait** that has two possible states, say a pea plant whose pods are either green or yellow. A pea plant has two genes for the trait (one from each parent), so the possible **genotypes** are

- gg, a gene for green pods from each parent.
- gy, a gene for green pods from one parent and a gene for yellow pods from the other (we usually cannot observe which parent contributed which gene).
- yy, a gene for yellow pods from each parent.

The genotypes gg and yy are called **homozygotes**, while the genotype gy is called **heterozygote**. Typically, one of the states of the inherited trait is **dominant** and the other **recessive**. Thus, for example, if green is the dominant state for pod color, then a plant with genotype gg or gy has green pods, while a plant with genotype yy has yellow pods. Genes are passed from parent to child in a random manner, so each new plant is a random experiment with respect to pod color.

Pod color in peas was actually one of the first examples of an inherited trait studied by Gregor Mendel, who is considered the father of modern genetics. Mendel also studied the color of the flowers (yellow or purple), the length of the stems (short or long), and the texture of the seeds (round or wrinkled).

A **sex-linked** hereditary disorder in humans is a disorder due to a defect on the **X chromosome** (one of the two chromosomes that determine gender). Suppose that n denotes the normal gene and d the defective gene linked to the disorder. Women have two X chromosomes, and typically d is **recessive**. Thus, a woman with genotype nn is completely normal with respect to the condition; a woman with genotype nd does not have the disorder, but is a **carrier**, since she can pass the defective gene to her children; and a woman with genotype dd has the disorder. A man has only one X chromosome (his other sex chromosome, the **Y chromosome**, typically plays no role in the disorder). A man with genotype n is normal and a man with genotype d has the disorder. Examples of sex-linked hereditary disorders are

dichromatism, the most common form of color-blindness, and the most common form of **hemophilia**, a bleeding disorder. Again, genes are passed from parent to child in a random manner, so the birth of a child is a random experiment in terms of the disorder.

Point Processes

There are a number of important processes that generate "random points in time". Often the random points are referred to as **arrivals**. Here are some specific examples:

- times that a piece of radioactive material emits elementary particles
- times that customers arrive at a store
- times that requests arrive at a web server
- failure times of a device

To formalize an experiment, we might record the number of arrivals during a specified interval of time or we might record the times of successive arrivals.

There are other processes that produce "random points in space". For example,

- flaws in a piece of sheet metal
- errors in a string of symbols (in a computer program, for example)
- raisins in a cake
- misprints on a page
- stars in a region of space

Again, to formalize an experiment, we might record the number of points in a given region of space.

Statistical Experiments

18. In 1879, Albert Michelson constructed an experiment for measuring the speed of light with an interferometer. The velocity of light data set contains the results of100 repetitions of Michelson's experiment. Explore the data set and explain, in a general way, the variability of the data.

19. In 1998, two students at the University of Alabama in Huntsville designed the following experiment: purchase a bag of M&Ms (of a specified advertised size) and record the counts for red, green, blue, orange, and yellow candies, and the net weight (in grams). Explore the M&M data. set and explain, in a general way, the variability of the data.

20. In 1999, two researchers at Belmont University designed the following experiment: capture a cicada in the Middle Tennessee area, and record the body weight (in grams), the wing length, wing width, and body length (in millimeters), the gender, and the species type. The cicada data set contains the results of 104 repetitions of this experiment. Explore the cicada data and explain, in a general way, the variability of the data.

21. On June 6, 1761, James Short made 53 measurements of the parallax of the sun, based on the transit of Venus. Explore the Short data set and explain, in a general way, the variability of the data.

22. In 1954, two massive field trials were conducted in an attempt to determine the effectiveness of the new vaccine developed by Jonas Salk for the prevention of polio. In both trials, a **treatment group** of children were given the vaccine while a **control group** of children were not. The incidence of polio in each group was measured. Explore the polio field trial data set and explain, in a general way, the underlying random experiment.

23. Each year from 1969 to 1972 a lottery was held in the US to determine who would be drafted for military service. Essentially, the lottery was a **ball and urn model** and became famous because many believed that the process was not sufficiently random. Explore the Vietnam draft lottery data set and speculate on how one might judge the degree of randomness.

Deterministic Versus Probabilistic Models

One could argue that some of the examples discussed above are inherently **deterministic**. In tossing a coin, for example, if we know the initial conditions (involving position, velocity, rotation, etc.), the forces acting on the coin (gravity, air resistance, etc.), and the makeup of the coin (shape, mass density, center of mass, etc.), then the laws of physics should allow us to predict precisely how the coin will land. This is true in a technical, theoretical sense, but false in a very real sense. Coins, dice, and many more complicated and important systems are **chaotic** in the sense that the outcomes of interest depend in a very sensitive way on the initial conditions and other parameters. In such situations, it might well be impossible to ever know the initial conditions and forces accurately enough to use deterministic methods.

In the coin experiment, for example, even if we strip away most of the real world complexity, we are still left with an essentially random experiment. Joseph Keller in a wonderful article "The Probability of Heads" deterministically analyzed the toss of a coin under a number of ideal assumptions:

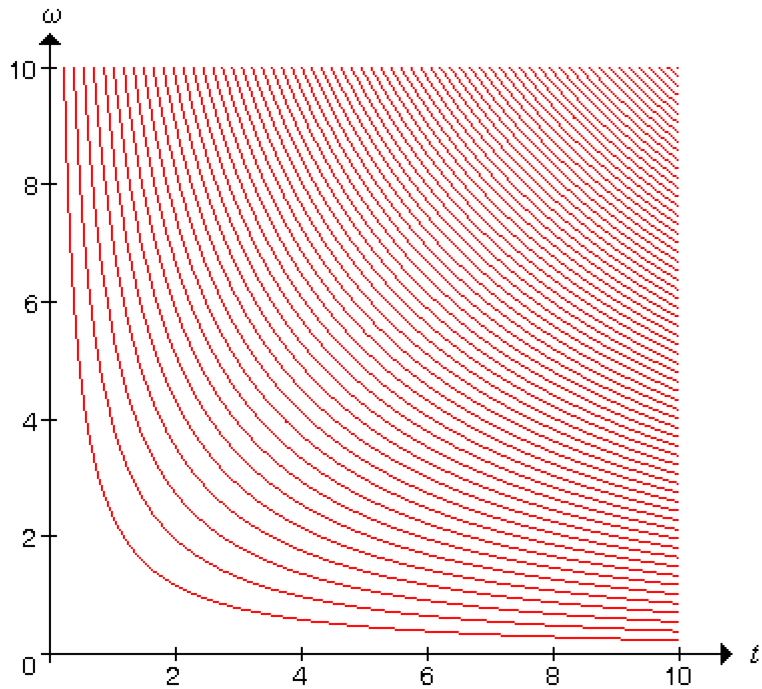
1. The coin is a perfect circle and has negligible thickness
2. The center of gravity of the coin is the geometric center.
3. The coin is initially heads up and is given an initial upward velocity u and angular velocity ω.
4. In flight, the coin rotates about a horizontal axis along a diameter of the coin.
5. In flight, the coin is governed only by the force of gravity. All other possible forces (air resistance or wind, for example) are neglected.
6. The coin does not bounce or roll after landing (as might be the case if it lands in sand or mud).

Of course, few of these ideal assumptions are valid for real coins tossed by humans. Let $t = \frac{u}{g}$ where g is the acceleration of gravity (in appropriate units). Note that the t just has units of time (in seconds) and hence is independent of how distance is measured. The scaled parameter t actually represents the time required for the coin to reach its maximum height.

Keller showed that the regions of the parameter space (t, ω) where the coin lands either heads up or tails up are separated by the curves

$$\omega = \left(2\,n \pm \frac{1}{2}\right)\frac{\pi}{2\,t}, \quad n \in \mathbb{N}$$

The parameter n is the total number of revolutions in the toss. A plot of some of these curves is given below. The largest region, in the lower left corner, corresponds to the event that the coin does not complete even one rotation, and so of course lands heads up, just as it started. The next region corresponds to one rotation, with the coin landing tails up. In general, the regions alternate between heads and tails.



The important point, of course, is that for even moderate values of t and ω, the curves are very close together, so that a small change in the initial conditions can easily shift the outcome from heads up to tails up or conversely. As noted in Keller's article, the renown probabilist and statistician Persi Diaconis determined experimentally that typical values of the initial conditions for a real coin toss are $t = \frac{1}{4}$ seconds and $\omega = 76\,\pi \approx 238.76$ radians per second. These values correspond to $n = 19$ revolutions in the toss. Of course, this parameter point is far beyond the region shown in our graph, in a region where the curves are exquisitely close together.