# 14. The Extreme Value Distribution

**Extreme value distributions** arise as limiting distributions for maximums or minimums (**extreme values**) of a sample of independent, identically distributed random variables, as the sample size increases. Thus, these distributions are important in statistics.

## The Standard Distribution for Maximums

### The Distribution Function

1. Show that the function given below is a distribution function for a continuous distribution on $\mathbb{R}$.

$$G(v) = e^{-e^{-v}}, \quad v \in \mathbb{R}$$

The distribution defined by the distribution function in Exercise 1 is the **type 1 extreme value distribution for maximums**. It is also known as the **Gumbel distribution** in honor of Emil Gumbel. This distribution arises as the limit of the maximum of $n$ independent random variables, each with the standard exponential distribution (when this maximum is appropriately scaled and centered). This is the main reason that the distribution is *special*, and is the reason for the name.

### The Density Function

2. Show that the density function is given by

$$g(v) = e^{-v} e^{-e^{-v}}, \quad v \in \mathbb{R}$$

3. Graph the density function and show that the distribution is unimodal and skewed right. In particular, show that

a. $g$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ and hence the mode occurs at 0
b. $g$ is concave upward on $(-\infty, -c)$ and on $(c, \infty)$, and concave downward on $(-c, c)$, where $c = \ln\left(\frac{3+\sqrt{5}}{2}\right)$.

4. In the random variable experiment, select the extreme value distribution and note the shape and location of the density function. Run the simulation 1000 times updating every 10 runs, and note the apparent convergence of the empirical density function to the probability density function.

### The Quantile Function

5. Show that the quantile function is

$$G^{-1}\left(p\right) = -\ln(-\ln(p)), \quad p \in (0, 1)$$

6. Show that

a. the first quartile is $-\ln(\ln(4)) \approx -0.3266$
b. the median is $-\ln(\ln(2)) \approx 0.3665$
c. the third quartile is $-\ln(\ln(4) - \ln(3)) \approx 1.2459$

7. In the quantile applet, select the extreme value distribution and note the shape and location of the density function and the distribution function. Compute the quantiles of order 0.1, 0.3, 0.6, and 0.9

**Moments**

The moment generating function of the standard extreme value distribution has a simple expression in terms of the gamma function.

8. Suppose that $V$ has the extreme value distribution for maximums. Show that the moment generating function is given by

$$m(t) = \mathbb{E}\left(e^{t\,V}\right) = \Gamma(1 - t), \quad t < 1$$

We can now compute the mean and variance. First, recall that the **Euler constant**, named for Leonhard Euler is defined by

$$\gamma = -\Gamma'(1) = -\int_0^\infty e^{-x} \ln(x) dx \approx 0.5772156649$$

9. Suppose that $V$ has the extreme value distribution for maximums. Show that

a. $\mathbb{E}(V) = \gamma$
b. $\mathrm{var}(V) = \frac{\pi^2}{6}$

10. In the random variable experiment, select the extreme value distribution and note the shape and location of the mean and standard deviation bar. Run the simulation 1000 times updating every 10 runs, and note the apparent convergence of the empirical moments to the true moments.

## The General Extreme Value Distribution

As with many other distributions we have studied, the standard extreme value distribution can be generalized by applying a linear transformation to the standard variable. Thus, suppose that $V$ has the type 1 extreme value distribution for maximums, discussed above. First, $U = -V$ has the **type 1 extreme value distribution for minimums**. More generally, we can form the location-scale family associated with these standard distributions. If $a \in \mathbb{R}$ and $b > 0$, then

- $X = a + b\,V$ has the **extreme value distribution for maximums** with **location parameter** $a$ and **scale parameter** $b$.
- $X = a - b\,V$ has the **extreme value distribution for minimums** with **location parameter** $a$ and **scale parameter** $b$.

**Distribution Functions**

11. Show that $X = a + b\,V$ has distribution function

$$F(x) = e^{-e^{-\frac{x-a}{b}}}, \quad x \in \mathbb{R}$$

.

12. Show that $X = a - b\,V$ has distribution function

$$F(x) = 1 - e^{-e^{\frac{x-a}{b}}}, \quad x \in \mathbb{R}$$

.

**Density Functions**

13. Show that $X = a + b\,V$ has density function

$$f(x) = \frac{1}{b}\, e^{-\frac{x-a}{b}}\, e^{-e^{-\frac{x-a}{b}}}, \quad x \in \mathbb{R}$$

.

14. Show that $X = a - b\,V$ has density function

$$f(x) = \frac{1}{b}\, e^{\frac{x-a}{b}}\, e^{-e^{\frac{x-a}{b}}}, \quad x \in \mathbb{R}$$

.

**Quantile Functions**

15. Show that $X = a + b\,V$ has quantile function

$$F^{-1}\left(p\right) = a - b\ln(-\ln(p)), \quad p \in (0,\ 1)$$

.

16. Show that $X = a - b\,V$ has quantile function

$$F^{-1}\left(p\right) = a + b\ln(-\ln(1 - p)), \quad p \in (0,\ 1)$$

.

**Moments**

17. Show that $X = a + b\,V$ has moment generating function

$$M(t) = e^{a\,t}\,\Gamma\left(1 - b\,t\right), \quad t < \frac{1}{b}$$

.

18. Show that $X = a - b\,V$ has moment generating function

$$M(t) = e^{a\,t}\,\Gamma\left(1 + b\,t\right), \quad t > -\frac{1}{b}$$

.

19. Show that

a. $\mathbb{E}(a + b\,V) = a + b\,\gamma$
b. $\mathbb{E}(a - b\,V) = a - b\,\gamma$
c. $\operatorname{var}(a + b\,V) = \operatorname{var}(a - b\,V) = b^2\,\frac{\pi^2}{6}$

**Transformations**

20. Show that

a. If $X$ has the standard exponential distribution then $U = \ln(X)$ has the standard extreme value distribution for minimums.
b. If $U$ has the standard extreme value distribution for minimums then $X = e^U$ has the standard exponential distribution.

21. More generally, show that

a. If $X$ has the Weibull distribution with shape parameter $k$ and scale parameter $b$ then $U = \ln(X)$ has the extreme value distribution for minimums, with location parameter $\ln(b)$ and scale parameter $\frac{1}{k}$.
b. If $U$ has the extreme value distribution for minimums, with location parameter $a$ and scale parameter $b$, then $X = e^U$ has the Weibull distribution with shape parameter $\frac{1}{b}$. and scale parameter $e^a$.