13. The Lognormal Distribution

A random variable X is said to have the **lognormal distribution** with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ if $\ln(X)$ has the normal distribution with mean μ and standard deviation σ. Equivalently, $X = e^Y$ where Y is normally distributed with mean μ and standard deviation σ. The lognormal distribution is used to model continuous random quantities when the distribution is believed to be skewed, such as certain income and lifetime variables.

Distribution

1. Use the change of variables theorem to show that the probability density function of the lognormal distribution with parameters μ and σ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma\,x}\exp\left(-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right), \quad x > 0$$

2. Show that the lognormal distribution is unimodal and skewed right. Specifically, let $m = \exp(\mu - \sigma^2)$ and show that

   a. $f\,x$ is increasing on $(0, m)$ and decreasing on (m, ∞), so that the mode occurs at $x = m$.
   b. $f(x) \to 0$ as $x \to \infty$.
   c. $f(x) \to 0$ as $x \downarrow 0$.

3. In the random variable experiment, select the lognormal distribution. Vary the parameters and note the shape and location of the density function. For selected values of the parameters, run the simulation 1000 times with an update frequency of 10. Note the apparent convergence of the empirical density to the true density.

Let Φ denote the standard normal distribution function. Recall that values of Φ are tabulated and can be obtained from the quantile applet, as well as standard mathematical and statistical software packages. Thus, the following exercises show how to compute the lognormal distribution function and quantiles in terms of the standard normal distribution function and quantiles.

4. Show that the lognormal distribution function F is given by

$$F(x) = \Phi\left(\frac{\ln(x)-\mu}{\sigma}\right), \quad x > 0$$

5. Show that the lognormal quantile function is given by

$$F^{-1}(p) = \exp\left(\mu + \sigma\,\Phi^{-1}(p)\right), \quad 0 < p < 1$$

6. Suppose that the income X of a randomly chosen person in a certain population (in \$1000 units) has the lognormal distribution with parameters $\mu = 2$ and $\sigma = 1$. Find $P(X > 20)$.

7. In the quantile applet, select the lognormal distribution. Vary the parameters and note the shape and location of the density function and the distribution function. With $\mu = 0$ and $\sigma = 1$, find the median and the first and third quartiles.

Moments

The moments of the lognormal distribution can be computed from the moment generating function of the normal distribution.

8. Suppose that X has the lognormal distribution with parameters μ and σ. Show that
$$\mathbb{E}(X^n) = \exp\left(n\,\mu + \frac{1}{2}n^2\,\sigma^2\right), \quad n \in \mathbb{N}$$

9. In particular, show that mean and variance of X are

   a. $\mathbb{E}(X) = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$
   b. $\text{var}(X) = \exp\left(2\left(\mu + \sigma^2\right)\right) - \exp\left(2\,\mu + \sigma^2\right)$

Even though the lognormal distribution has finite moments of all orders, the moment generating function is infinite at any positive number. This property is one of the reasons for the fame of the lognormal distribution.

10. Show that $\mathbb{E}\left(e^{t\,X}\right) = \infty$ for any $t > 0$.

11. Suppose that the income X of a randomly chosen person in a certain population (in $1000 units) has the lognormal distribution with parameters $\mu = 2$ and $\sigma = 1$. Find each of the following:

   a. $\mathbb{E}(X)$
   b. $\text{sd}(X)$

12. In the simulation of the random variable experiment, select the lognormal distribution. Vary the parameters and note the shape and location of the mean/standard deviation bar. For selected values of the parameters, run the simulation 1000 times with an update frequency of 10. Note the apparent convergence of the empirical moments to the true moments.

Transformations

The most important transformations are the ones in the definition: if X has a lognormal distribution then $\ln(X)$ has a normal distribution; conversely if Y has a normal distribution then e^Y has a lognormal distribution.

13. For fixed σ, show that the lognormal distribution with parameters μ and σ is a scale family with scale parameter e^μ.

14. Show that the lognormal distribution is a 2-parameter exponential family with natural parameters and natural

statistics, respectively, given by

a. $\left(-\dfrac{1}{2\,\sigma^2},\ \dfrac{\mu}{\sigma^2}\right)$

b. $\left(\ln(X)^2,\ \ln(X)\right)$