11. The Pareto Distribution

The Pareto distribution is a skewed, heavy-tailed distribution that is sometimes used to model the distribution of incomes.

The Basic Pareto Distribution

1. Let $a > 0$ be a parameter. Show that the function F given below is a distribution function.

$$F(x) = 1 - \frac{1}{x^a}, \quad x \geq 1$$

The distribution defined by the function in Exercise 1 is called the **Pareto distribution** with **shape parameter** a, and is named for the economist Vilfredo Pareto.

2. Show that the probability density function f is given by

$$f(x) = \frac{a}{x^{a+1}}, \quad x \geq 1$$

3. Sketch the graph of the probability density function f. Note that f is decreasing, so in particular, the mode occurs at $x = 1$ for any a. Of course, f decreases faster as a increases.

4. In the simulation of the random variable experiment, select the Pareto distribution. Vary the shape parameter and note the shape and location of the density function. For selected values of the parameter, run the simulation 1000 times with an update frequency of 10 and note the apparent convergence of the empirical density to the true density.

5. Show that the quantile function is

$$F^{-1}\left(p\right) = \frac{1}{(1-p)^{\frac{1}{a}}}, \quad 0 < p < 1$$

6. Find the median and the first and third quartiles for the Pareto distribution with shape parameter $a = 3$. Compute the interquartile range.

7. In the quantile applet, select the Pareto distribution. Vary the shape parameter and note the shape and location of the density function and the distribution function.

The Pareto distribution is a **heavy-tailed** distribution. Thus, the mean, variance, and other moments are finite only if the shape parameter a is sufficiently large.

8. Suppose that X has the Pareto distribution with shape parameter $a > 0$. Show that

$$\mathbb{E}\left(X^n\right) = \begin{cases} \frac{a}{a-n}, & 0 < n < a \\ \infty, & n \geq a \end{cases}$$

9. Use the result of the previous exercise to show that

a. $\mathbb{E}(X) = \frac{a}{a-1}$ if $a > 1$
b. $\text{var}(X) = \frac{a}{(a-1)^2 (a-2)}$ if $a > 2$

10. In the random variable experiment, select the Pareto distribution. Vary the parameters and note the shape and location of the mean/standard deviation bar. For each of the following parameter values, run the simulation 1000 times with an update frequency of 10 and note the behavior of the empirical moments:

a. $a = 1$
b. $a = 2$
c. $a = 3$

The General Pareto Distribution

As with many other distributions, the Pareto distribution is often generalized by adding a scale parameter. Thus, suppose that Z has the basic Pareto distribution with shape parameter a. If $b > 0$, the random variable $X = b Z$ has the **Pareto distribution** with **shape parameter** a and **scale parameter** b. Note that X takes values in the interval $[b, \infty]$.

Analogies of the results given above follow easily from basic properties of the scale transformation.

11. Show that the probability density function is

$$f(x) = \frac{a b^a}{x^{a+1}}, \quad x \geq b$$

12. Show that the distribution function is

$$F(x) = 1 - \left(\frac{b}{x}\right)^a, \quad x \geq b$$

13. Show that the quantile function is

$$F^{-1}(p) = \frac{b}{(1-p)^{1/a}}, \quad 0 < p < 1$$

14. Show that the moments are given by

$$\mathbb{E}(X^n) = \begin{cases} b^n \frac{a}{a-n}, & 0 < n < a \\ \infty, & n \geq a \end{cases}$$

15. Show that the mean and variance are

a. $\mathbb{E}(X) = b \frac{a}{a-1}$ if $a > 1$
b. $\text{var}(X) = b^2 \frac{a}{(a-1)^2 (a-2)}$ if $a > 2$

16. Suppose that the income of a certain population has the Pareto distribution with shape parameter 3 and scale parameter 1000.

a. Find the proportion of the population with incomes between 2000 and 4000.
b. Find the median income.
c. Find the first and third quartiles and the interquartile range.
d. Find the mean income.
e. Find the standard deviation of income.
f. Find the 90th percentile.

Transformations

The following exercise is a restatement of the fact that b is a scale parameter.

17. Suppose that X has the Pareto distribution with shape parameter a and scale parameter b. Show that if $c > 0$ then $c\,X$ has the Pareto distribution with shape parameter a and scale parameter $b\,c$

18. Suppose that X has the basic Pareto distribution with shape parameter a. Show that $\frac{1}{X}$ has the beta distribution with left parameter a and right parameter 1.