10. The Zeta Distribution

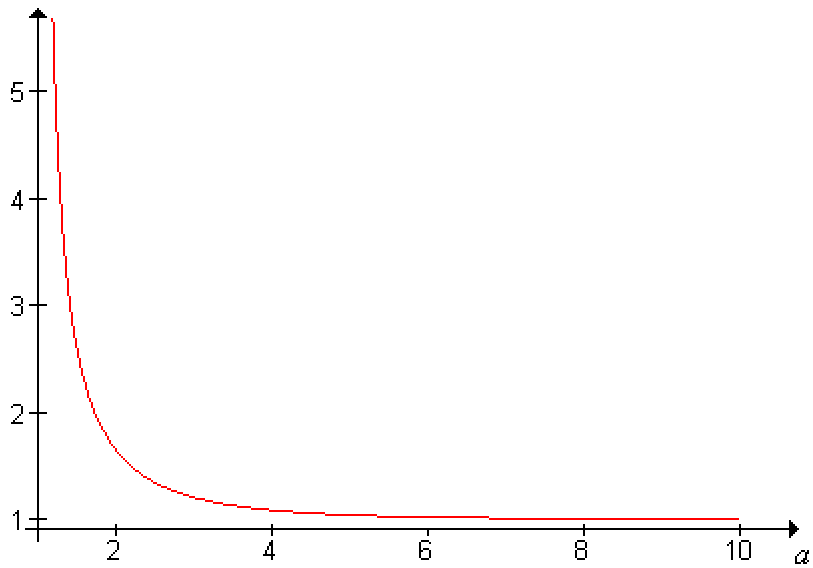
The **zeta distribution** is used to model the size or ranks of certain types of objects randomly chosen from certain types of populations. Typical examples include the frequency of occurrence of a word randomly chosen from a text, or the population rank of a city randomly chosen from a country. The zeta distribution is also known as the **Zipf distribution**, in honor of the American linguist George Zipf.

The Zeta Function

The **Riemann zeta function** ζ, named after Bernhard Riemann, is defined as follows:

$$\zeta(a) = \sum_{n=1}^{\infty} \frac{1}{n^a}, \quad a > 1$$

(You might recall from calculus that the series in the zeta function converges for $a > 1$ and diverges for $a \le 1$. A graph of the zeta function on the interval $(1, 10]$ is given below:



1. Try to verify the main properties of the graph analytically. In particular, show that

a. ζ is decreasing.
b. ζ is concave upward.
c. $\zeta(a) \downarrow 1$ as $a \uparrow \infty$
d. $\zeta(a) \uparrow \infty$ as $a \downarrow 1$

The zeta function is transcendental, and most of its values must be approximated. However, $\zeta(a)$ can be given explicitly for even integer values of a; in particular, $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$.

The Probability Density Function

2. Show that the function f given below is probability density function for any $a > 1$.

$$f(n) = \frac{1}{\zeta(a)\, n^a}, \quad n \in \mathbb{N}_+$$

The discrete distribution defined by the density function in Exercise 2 is called the **zeta distribution** with parameter a. In an algebraic sense, the zeta distribution is a discrete version of the Pareto distribution.

3. Let X denote the frequency of occurrence of a word chosen at random from a certain text, and suppose that X has the zeta distribution with parameter $a = 2$. Find $\mathbb{P}(X > 4)$.

4. Suppose that X has the zeta distribution with parameter a. Show that the distribution is a one-parameter exponential family with natural parameter a and natural statistic $-\ln(X)$.

Moments

The moments of the zeta distribution can be expressed easily in terms of the zeta function.

5. Suppose that X has the zeta distribution with parameter a and that $k \ge 0$. Show that

$$\mathbb{E}\left(X^k\right) = \begin{cases} \infty, & a \le k + 1 \\ \dfrac{\zeta(a-k)}{\zeta(a)}, & a > k + 1 \end{cases}$$

6. In particular, show that

a. $\mathbb{E}(X) = \dfrac{\zeta(a-1)}{\zeta(a)}$ if $a > 2$.
b. $\operatorname{var}(X) = \dfrac{\zeta(a-2)}{\zeta(a)} - \left(\dfrac{\zeta(a-1)}{\zeta(a)}\right)^2$ if $a > 3$.

7. Let X denote the frequency of occurrence of a word chosen at random from a certain text, and suppose that X has the zeta distribution with parameter $a = 4$. Approximate each of the following:

a. $\mathbb{E}(X)$.
b. $\operatorname{var}(X)$.