DIPLOMA THESIS

Stochastic Investigation of Random Walks with Internal States

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1 Introduction

In the last century, a lot of interest and discussion was incited by the theory of random walks (see [12], for instance). The reason for it is clear: despite the simplicity of the model, a vast number of profound features and interesting phenomena can be investigated. In the last decades, with the development of mathematical physics, Lorentz processes came to the limelight. The model of random walk with internal states (or, alternatively, random walk with internal degrees of freedom; briefly RWwIS) was introduced by Sinai in 1981 in his Kyoto talk [13]. His aim was to get an efficient tool for examining the Lorentz process (in this context, internal states would represent the elements of the Markov partition), but other applications can be found, for instance, in some models of queueing systems. However, the investigation of this model is important, as it is a manifest generalization of simple symmetric random walk. Let us begin with the definition of RWwIS (with the notation in [8]).

Definition 1 *Let E be a finite set. On the set $H = \mathbb{Z}^d \times E$ $(d = 1, 2, ...)$, the Markov chain $\xi_n = (\eta_n, \varepsilon_n)$ is a random walk with internal states (RWwIS), if for $\forall x_n, x_{n+1} \in \mathbb{Z}^d, \quad j_n, j_{n+1} \in E$*

$$P(\xi_{n+1} = (x_{n+1}, j_{n+1}) | \xi_n = (x_n, j_n)) = p_{x_{n+1}-x_n, j_n, j_{n+1}}.$$

In fact, E could be countable, as well, but we will consider only the finite case. We will denote $s = \#E$.

There are some basic assumptions which will always be supposed. These are the following:

(i) $(\varepsilon_0, \varepsilon_1, ...)$ - obviously a Markov chain - is irreducible and aperiodic (its stationary distribution will be denoted by μ)

(ii) the arithmetics are trivial, with the notation in [8], $L = \mathbb{Z}^d$

(iii) the expectation of one step is zero provided that ε is distributed according to its unique stationary measure

(iv) the covariance matrix, which is exactly defined in Section 2, exists and is nonsingular.

In general, we will assume that $\eta_0 = 0$.

Let us introduce some notations. Let $L_d(n)$ denote the number of distinct sites visited by a RWwIS up to n steps. The expectation of $L_d(n)$ is $E_d(n)$, and the variance is $V_d(n)$. Our first goal (Section 3 and Section 4) is to prove theorems corresponding to the ones in [3], concerning the same quantities of simple symmetric random walks (which will later on be referred to as SSRW).

We have some figures showing trajectories of some random walks. Figure 1 shows a random trajectory of a two dimensional RWwIS. Black points are the sites visited during the first 100 steps. The red point shows the place where the wandering particle is situated at step 100. In this case, $L_2(100) = 50$. Figure 1 shows a trajectory of the first 22 steps of a three dimensional RWwIS. The meaning of the black and red points is the same as in the case of Figure 1. In this case, $L_3(22) = 16$.

After proving our Dvoretzky-Erdős type theorems, some other interesting properties of our model will be discussed.

My diploma thesis is organized as follows: in Section 2 the main theorem of [8] is generalized. Namely, a remainder term of the local limit theorem is computed, as it will be necessary for estimating $E_2(n)$. A further refinement of the local limit theorem will also be given as it will be useful when proving the strong law of large numbers in the plane. In Section 3, the number of visited points in the high dimensional case, i.e. when $d \geq 3$, is dealt with. We prove asymptotics for $E_d(n)$, and estimate $V_d(n)$, from which we can prove both the weak and strong laws of large numbers. In this Section, we will not use the result of Section 2, Theorem 5.2. in [8] will be enough for our purposes. In Section 4 the $d = 2$ case is discussed. For $E_2(n)$, same asymptotics ($const \frac{n}{\log n}$) is found as in [3], but with some different constant. $V_2(n)$ is also estimated, and the weak law of large numbers is also proved. The proof of the strong law in the plane is a little bit cumbersome calculation, so it is postponed to Section 5. This result will not be used in the sequel. In Section 6, the one dimensional settings are considered. This case requires a little bit different approach from the previous ones, so application of a Tauberian theorem will be very useful.

In Section 7, we are concerned with the order of decay of the autocorrelation function of some locally perturbed RWwIS. A related question is the asymptotic order of the probability of the first return to the origin at time n. Surprisingly, I did not manage

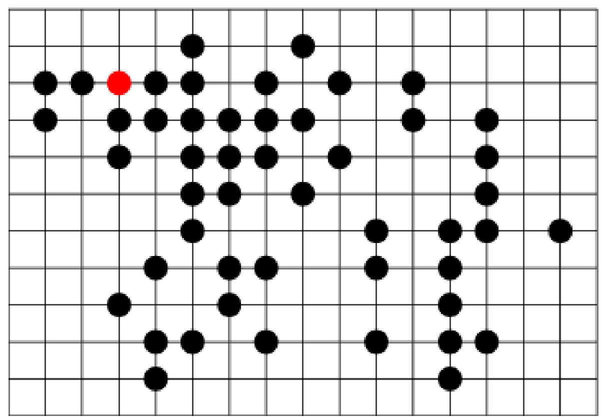
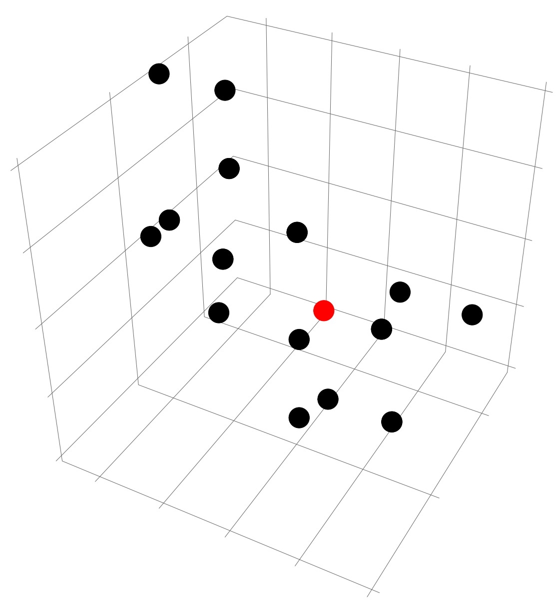


Figure 1: The first 100 steps of a RWwIS in $d = 2$

to find any theorem concerning this question in $d \geq 2$, so Theorem 9 in Section 7 is new in the case of SSRW, as well. Section 8 deals with the distribution of the internal states at the time of reaching the origin. Here we do not have many rigorous results, so examples and simulations are used for illustration. In Section 9, some other interesting path properties inspired mainly by [4] are discussed. What computed here is the number of returns to the origin in the general case $d \geq 2$, and the number of distinct sites visited only once up to time n in two specific examples. Section 10 gives conclusions and some remarks.



The first 22 steps of a RWwIS in $d = 3$

2 Local limit theorem with remainder term

In this section, we begin with calculating a remainder term for Theorem 5.2. in [8]. Furthermore, another refinement of this theorem will be proved, as it will be used when proving the strong law of large numbers in the plane. Finally, the reversed walk will be defined. First, we reformulate the mentioned theorem. We have to start with some definitions. Denote

$$\begin{aligned} A_y &= (p_{y,j,k})_{j,k=1,\dots,s} : \mathbb{C}^s \to \mathbb{C}^s, \\ Q &= \sum_{y\in\mathbb{Z}^d} A_y, \\ M_l &= \sum_{y\in\mathbb{Z}^d} y_l A_y, \\ \Sigma_{l,m} &= \sum_{y\in\mathbb{Z}^d} y_l y_m A_y. \end{aligned}$$

So, the transition matrix of the Markov chain $(\varepsilon_0, \varepsilon_1, ...)$ is Q and its unique stationary measure is μ.

Theorem 1 *If a RWwIS in $\mathbb{Z}^d$ fulfills our basic assumptions and the matrix*

$$\sigma = (\sigma_{l,m})_{1\le l,m\le d}$$

whose elements are

$$\sigma_{l,m} = \langle \mu, \Sigma_{l,m} 1\rangle - \left\langle \mu, M_l (Q-1)^{-1} M_m 1\right\rangle - \left\langle \mu, M_m (Q-1)^{-1} M_l 1\right\rangle$$

(which can be called a covariance matrix) is positive definite, then

$$\sum_{(x,k)\in H} \left| P\left(\xi_n = (x,k) | \xi_0 = (0,j)\right) - n^{-d/2}\mu_k g_\sigma\left(\frac{x}{\sqrt{n}}\right)\right| \to 0$$

as $n\to\infty$, where $g_\sigma(x)$ denotes the density of a Gaussian distribution with mean 0 and covariance matrix σ.

Of course, the condition concerning the positive definiteness of the matrix in one dimension means $\sigma > 0$. We omit the proof, it can be found in [3]. In fact, there is a typo in [3] as they write $n^{-1/2}$ instead of $n^{-d/2}$ but it is easy to correct it even in the proof.

Our calculation will be similar to the one of [8]. The main point is that while in [8] it is sufficient to consider the Taylor expansion of the largest eigenvalue up to the quadratic term, now, we have to calculate the third term, as well. We have to introduce some notations.

Let $\alpha(t) = \sum_{y \in \mathbb{Z}^d} \exp\left(i \langle t, y \rangle\right) A_y,\ t \in [-\pi, \pi]^d$. Now, we have to consider the Taylor expansion of the largest eigenvalue of $\alpha(t)$, which is denoted by $\lambda(t)$, up to the third term.

Let us first assume that $d = 1$. From our basic assumptions it follows that $M = \sum_{y \in \mathbb{Z}} y A_y$ and $\Sigma = \sum_{y \in \mathbb{Z}} y^2 A_y$ are convergent series. But for know, we suppose the convergence of

$$\Xi = \sum_{y \in \mathbb{Z}} y^3 A_y. \tag{1}$$

The existence of M, Σ and Ξ implies

$$\alpha(t) = Q + itM - \frac{t^2}{2}\Sigma - \frac{it^3}{6}\Xi + o(t^3) \quad (t \to 0). \tag{2}$$

Now, by perturbation theoretic means (i.e. the straightforward extension of Theorem 5.11. of Chapter II. in [6]) it can be easily proved that

$$\lambda(t) = 1 + r_1 t + \frac{r_2}{2}t^2 + \frac{r_3}{6}t^3 + o(t^3) \quad (t \to 0). \tag{3}$$

From [8] we know that $r_1 = 0$ and $r_2 = -\langle \Sigma 1, \mu \rangle + 2\langle M(Q-1)^{-1}M, \mu \rangle$.

For the computation of r_3 we use the same method as for r_1 and r_2. Let $\Pi : \mathbb{C}^s \to \mathbb{C}^s \quad \Pi\Psi = \langle \Psi, \mu \rangle 1$ and $B = (Q - 1 + c\Pi)^{-1}$ for some real $c \neq 0$. Then

$$\begin{aligned} & (\alpha(t) - \lambda(t) + c\Pi)^{-1} \\ = & \left(Q + itM - \frac{t^2}{2}\Sigma - \frac{it^3}{6}\Xi - 1 - r_1 t - \frac{r_2}{2}t^2 - \frac{r_3}{6}t^3 + c\Pi + o(t^3)\right)^{-1} \\ = & \left(1 + itBM - \frac{t^2}{2}B\Sigma - \frac{it^3}{6}B\Xi - r_1 tB - \frac{r_2}{2}t^2 B - \frac{r_3}{6}t^3 B + o(t^3)\right)^{-1} B \\ = & S^{-1}B. \end{aligned}$$

Now, elementary calculations show that

$$\begin{aligned} S = \; & B - itBMB + \frac{t^2}{2}B\Sigma B + \frac{it^3}{6}B\Xi B + \frac{r_2}{2}t^2 B^2 + \frac{r_3}{6}t^3 B^2 \\ & - t^2 BMBMB - \frac{it^3}{2}B\Sigma BMB - \frac{it^3}{2}BMB\Sigma B - \frac{ir_2}{2}t^3 BMB^2 \\ & + it^3 BMBMBMB + o(t^3). \end{aligned}$$

Using $B1 = c^{-1}1$, $B^*\mu = c^{-1}\mu$ and $1 = c\left\langle (\alpha(t) - \lambda(t) + c\Pi)^{-1}1, \mu \right\rangle$ we conclude

$$r_3 = i\left(3\left\langle \Sigma BM1, \mu\right\rangle + 3\left\langle MB\Sigma 1, \mu\right\rangle - \left\langle \Xi 1, \mu\right\rangle\right).$$

Using the notation $\sigma^2 = -r_2$ we can now formulate our theorem:

Theorem 2 *Our basic assumptions and the existence of (1) imply:*

$$\begin{aligned} P(\xi_n &= (x,k)|\xi_0 = (0,j)) - \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{x^2}{2n\sigma^2}\right)\left[1 - \frac{ir_3}{6}x\left(3\sigma^2 n - x^2\right)\frac{1}{\sigma^6}\frac{1}{n^2}\right] \\ &= o\left(\frac{1}{n}\right), \end{aligned}$$

where the small order is uniform in x.

Proof. The proof is similar to the one of Theorem 2.1. in [8]. Because of (3) we have

$$\alpha^n(t) = \langle 1\mu^*\rangle \left(1 - \frac{\sigma^2 t^2}{2} + \frac{r_3}{6}t^3 + o\left(t^3\right)\right)^n (1 + o(1)), \tag{4}$$

where $o(1)$ on the right hand side is the contribution of the other eigenvalues besides $\lambda(t)$. Thus this $o(1)$ can be neglected, when substituting $s/\sqrt{n}$ to t, as it yields an exponentially small term and we prove polynomial error term. Elementary calculations show that

$$\left(1 - \frac{\sigma^2 s^2}{2n} + \frac{r_3}{6}\frac{s^3}{n^{\frac{3}{2}}} + o\left(\frac{s}{n^{\frac{3}{2}}}\right)\right)^n = \exp\left(-\frac{\sigma^2 s^2}{2}\right)\left(1 + \frac{r_3}{6}s^3\frac{1}{\sqrt{n}} + o\left(\frac{s^3}{\sqrt{n}}\right)\right). \tag{5}$$

In order to prove the statement, we use the Fourier transforms and the usual estimations

$$\begin{aligned} &\left\| \sqrt{n}\int_{-\pi}^{\pi} \exp(-ixt)\, e_j^* \alpha^n(t)\, dt - \mu^* \frac{\sqrt{2\pi}}{\sigma}\exp\left(-\frac{x^2}{2n\sigma^2}\right)\left[1 - \frac{ir_3}{6}x\left(3\sigma^2 n - x^2\right)\frac{1}{\sigma^6}\frac{1}{n^2}\right]\right\| \\ \leq &\int_{|s|<n^\varepsilon} \left\| e_j^*\alpha^n\left(\frac{s}{\sqrt{n}}\right) - \mu^*\exp(-\frac{\sigma^2 s^2}{2})\left(1 + \frac{r_3}{6}\frac{s^3}{\sqrt{n}}\right)\right\| ds \\ &+ \|\mu\| \int_{|s|>n^\varepsilon} \exp(-\frac{\sigma^2 s^2}{2}) ds + \int_{n^\varepsilon<|s|<\gamma\sqrt{n}} \left\| e_j^*\alpha^n\left(\frac{s}{\sqrt{n}}\right)\right\| ds \\ &+ \int_{\gamma\sqrt{n}<|s|<\pi\sqrt{n}} \left\| e_j^*\alpha^n\left(\frac{s}{\sqrt{n}}\right)\right\| ds \\ = &\; I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where $0 < \varepsilon < \frac{1}{6}$ is arbitrary. It is clear that proving $I_j = o\left(\frac{1}{\sqrt{n}}\right), \quad j = 1,2,3,4$ is enough for our purposes. (4) and (5) yield that the integrand in I_1 is equal to $\frac{\delta(n)}{n^{1/2}} s^3 \exp\left(-\frac{\sigma^2 s^2}{2}\right)$, where $\delta(n) \to 0$ uniformly in s. Thus we have $I_1 = o\left(\frac{1}{\sqrt{n}}\right)$. It is clear that $I_2 = o\left(\frac{1}{\sqrt{n}}\right)$, and I_4 converges exponentially fast to zero. Now, we have to estimate I_3. It is easy to see that (2) yields $\|\alpha(t)\| \leq \exp\left(-\frac{\sigma^2 t^2}{4}\right)$, if $|t|$ is smaller than an appropriate $\gamma > 0$ constant. Thus

$$I_3 = \int\limits_{n^\varepsilon < |s| < \gamma\sqrt{n}} \left\| e_j^* \alpha^n \left(\frac{s}{\sqrt{n}}\right) \right\| ds = \sqrt{n} \int\limits_{n^{\varepsilon - 1/2} < |t| < \gamma} \left\| e_j^* \alpha^n (t) \right\| dt \leq$$

$$\leq \sqrt{n} \int\limits_{n^{\varepsilon - 1/2} < |t| < \gamma} \exp\left(-\frac{\sigma^2 t^2 n}{4}\right) dt = \int\limits_{n^\varepsilon < |s| < \gamma\sqrt{n}} \exp\left(-\frac{\sigma^2 s^2}{4}\right) ds.$$

So we have $I_3 = o\left(\frac{1}{\sqrt{n}}\right)$, too. ■

Remark 1 *In Theorem 2 for the expression subtracted from the appropriate probability we have:*

$$\mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{x^2}{2n\sigma^2}\right) \left[1 - \frac{ir_3}{6} x \left(3\sigma^2 n - x^2\right) \frac{1}{\sigma^6} \frac{1}{n^2}\right]$$

$$= \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{y^2}{2}\right) + \mu_k \frac{1}{\sqrt{n}\sigma} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \frac{ir_3}{6} \left(y^3 n^{3/2} \sigma^3 - 3y n^{3/2} \sigma^3\right) \frac{1}{\sigma^6} \frac{1}{n^2}$$

$$= \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{y^2}{2}\right) + \mu_k \frac{1}{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \frac{ir_3}{6} \left(y^3 - 3y\right) \frac{1}{\sigma^4}$$

$$= \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{y^2}{2}\right) + \mu_k \frac{1}{\sqrt{n}} \frac{1}{\sigma} \frac{q_1(y)}{\sqrt{n}},$$

where $y = \frac{x}{\sqrt{n}\sigma}$, *and the* $q_1(y)$ *is the function defined in [9], Chapter VI. (1.14.). In this sense, the local limit theorem concerning RWwIS is analogous to the one of simple symmetric random walk (see [9] Chapter VII. Theorem 13).*

The extension of Theorem 2 to the multidimensional case is straightforward. Analogously to (3), we have:

$$\lambda(t) = 1 - \frac{1}{2} t^T \sigma t + f(t) + o(|t|^3) \quad (|t| \to 0),$$

where $f(t) = \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} r_{3,i,j,k} t_i t_j t_k$ is the third term of the Taylor expansion. Denote

$$\Omega = \frac{n^{d/2}}{(2\pi)^d} P(\xi_n = (x,.) | \xi_0 = (0,j)) = \frac{n^{d/2}}{(2\pi)^d} \int\limits_{-\pi}^{\pi} \ldots \int\limits_{-\pi}^{\pi} \exp\left(-i \langle x, t \rangle\right) e_j^* \alpha^n (t)\, dt.$$

So the analogue of the expression subtracted from the appropriate probability in Theorem 2 (multiplied by $\frac{n^{d/2}}{(2\pi)^d}$) is

$$I^{(n)} := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp\left(-\frac{s\sigma s}{2} - i\left\langle x, \frac{s}{\sqrt{n}}\right\rangle\right) \frac{f(s)}{\sqrt{n}} ds.$$

Using Lebesgue's Theorem, it is easy to see that $I^{(n)} = O\left(n^{-1/2}\right)$. One can estimate I_1, I_2, I_3, I_4 the same way, as it was done in the proof of Theorem 2 (see [8] Section 5. for more details). So we have arrived at

Proposition 1 *For the d dimensional RWwIS we have*

$$P(\xi_n = (x, k)|\xi_0 = (0, j)) = \frac{1}{n^{d/2}} \mu_k g_\sigma \left(\frac{x}{\sqrt{n}}\right) + O\left(n^{-(d+1)/2}\right).$$

where $g_\sigma(x)$ denotes the density of a Gaussian distribution with mean 0 and covariance matrix σ.

A further refinement of the local limit theorem will be useful in the sequel. Now, we would like to go further in the asymptotic expansion, and apply our techniques in the two dimensional case. Nevertheless, we are interested only in an estimation, not in the exact result which will simplify the calculation. Just like previously, let us begin with the one dimensional case. Assume the convergence of the series

$$\Upsilon = \sum_{y \in \mathbb{Z}} y^4 A_y. \tag{6}$$

Now, just like previously, we may write

$$\alpha(t) = Q + itM - \frac{t^2}{2}\Sigma - \frac{it^3}{6}\Xi + \frac{t^4}{24}\Upsilon + o(t^4) \quad (t \to 0)$$

for the Fourier transform, and

$$\lambda(t) = 1 + r_1 t - \frac{\sigma^2}{2} t^2 + \frac{r_3}{6} t^3 + O(t^4) \quad (t \to 0) \tag{7}$$

for the largest eigenvalue of $\alpha(t)$. As previously, we have

$$\left(1 - \frac{\sigma^2 s^2}{2n} + \frac{r_3}{6}\frac{s^3}{n^{\frac{3}{2}}} + O\left(\frac{s^4}{n^2}\right)\right)^n = \exp\left(-\frac{\sigma^2 s^2}{2}\right)\left(1 + \frac{r_3}{6} s^3 \frac{1}{\sqrt{n}} + O\left(\frac{s^3 + s^4}{n}\right)\right).$$

A very similar argument to the previous one (with $I_j = o\left(\frac{1}{n}\right), \quad j = 1, 2, 3, 4$) leads to

$$\begin{aligned} P(\xi_n &= (x,k)|\xi_0 = (0,j)) \qquad (8)\\ &= \mu_k \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{x^2}{2n\sigma^2}\right)\left[1 - \frac{ir_3}{6} x \left(3\sigma^2 n - x^2\right) \frac{1}{\sigma^6} \frac{1}{n^2}\right] + O\left(\frac{1}{n^{3/2}}\right), \end{aligned}$$

where the great order on the right hand side is uniform in x.

Now our aim is to formulate an assertion similar to (8) in two dimensions. Applying the one dimensional proof to the two dimensional case it is easily seen that

$$P(\xi_n = (x,k)|\xi_0 = (0,j)) - \mu_k \frac{1}{n} g_\sigma \left(\frac{x}{\sqrt{n}}\right) + F(x,n) = O\left(\frac{1}{n^2}\right),$$

where the great order is again uniform in x, and

$$F(x,n) = \frac{1}{2\pi n^{3/2}} \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \sum_{i_3=1}^{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{s^T \sigma s}{2} - i\left\langle x, \frac{s}{\sqrt{n}}\right\rangle\right) r_{3,i_1,i_2,i_3} s_{i_1} s_{i_2} s_{i_3} ds.$$

We estimate $F(x,n)$ just like it was done in [10]. Observe that with the notation

$$\Psi(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{s^T \sigma s}{2} - i\langle x, s\rangle\right) ds = \frac{2\pi}{\sqrt{|\sigma|}} \exp\left(-\frac{x^T \sigma^{-1} x}{2}\right),$$

we have with an appropriate C_1 constant

$$|F(x,n)| < C_1 \frac{1}{n^{3/2}} \max_{i_1,i_2,i_3} \left| \frac{\partial^3 \Psi}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}} \left(\frac{x}{\sqrt{n}}\right)\right|.$$

Further, observe that

$$\left| \frac{\partial^3 \Psi}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}} (x)\right| < C_2 \left(\|x\| + \|x\|^3\right) \exp\left(-\frac{x^T \sigma^{-1} x}{2}\right).$$

So we have arrived at

Proposition 2 *For a RWwIS fulfilling our basic assumptions and for which (6) exists there is a C constant, such that for every $x \in \mathbb{R}^2$ and for every $1 \le j,k \le s$ the following estimation holds*

$$\begin{aligned} &\left| P(\xi_n = (x,k)|\xi_0 = (0,j)) - \mu_k \frac{1}{n} g_\sigma\left(\frac{x}{\sqrt{n}}\right)\right| \\ \le\ & C\left(\frac{1}{n^{3/2}}\left(\frac{\|x\|}{n^{1/2}} + \frac{\|x\|^3}{n^{3/2}}\right) \exp\left(-\frac{x^T \sigma^{-1} x}{2n}\right) + \frac{1}{n^2}\right). \end{aligned}$$

The so-called reversed walk will be important in the sequel. If a RWwIS is given with the appropriate $(p_{y,i,j})$ probabilities, then we define the $(q_{y,i,j})$ reversed random walk for which

$$q_{y,i,j} = \frac{\mu_j p_{-y,j,i}}{\mu_i}. \tag{9}$$

Obviously, the stationary measure of the reversed walk is also μ. As we would like to apply the local limit theorem for the reversed walk, we need

Proposition 3 *If the primary RWwIS fulfills our basic assumptions, then the reversed walk fulfills them as well. Furthermore, the so-called covariance matrix of the reversed walk is the same as the one of the primary walk.*

Proof. Basic assumptions (i)-(iii) are fulfilled obviously. So it suffices to prove the second statement. Let us introduce some notations

$$\begin{aligned}
\widetilde{A}_y &= (q_{y,j,k})_{j,k=1,\ldots,s}, \\
\widetilde{Q} &= \sum_{y\in\mathbb{Z}^d} \widetilde{A}_y, \\
\widetilde{M}_l &= \sum_{y\in\mathbb{Z}^d} y_l \widetilde{A}_y, \\
\widetilde{\Sigma}_{l,m} &= \sum_{y\in\mathbb{Z}^d} y_l y_m \widetilde{A}_y,
\end{aligned}$$

and a new inner product

$$\begin{aligned}
(,) &: \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}, \\
(u,v) &= \sum_{i=1}^{s} \mu_i u_i v_i.
\end{aligned}$$

Let us denote by A^* the adjoint of the linear operator A, i.e. $(u, Av) = (A^* u, v)$ for all $u, v \in \mathbb{R}^s$. Elementary calculations show that $\widetilde{Q} = Q^*$, $\widetilde{A}_y = (A_{-y})^*$, $\widetilde{M}_l = -(M_l)^*$, $\widetilde{\Sigma}_{l,m} = (\Sigma_{l,m})^*$ for all $y \in \mathbb{Z}^d, 1 \le l, m \le s$. Now, for an arbitrary element $\widetilde{\sigma}_{l,m}$ of the "covariance matrix" defined for the reversed walk

$$\begin{aligned}
\widetilde{\sigma}_{l,m} &= \left(1, \widetilde{\Sigma}_{l,m} 1\right) - \left(1, \widetilde{M}_l \left(\widetilde{Q} - 1\right)^{-1} \widetilde{M}_m 1\right) - \left(1, \widetilde{M}_m \left(\widetilde{Q} - 1\right)^{-1} \widetilde{M}_l 1\right) \\
&= (\Sigma_{l,m} 1, 1) - \left(M_m (Q-1)^{-1} M_l 1, 1\right) - \left(M_l (Q-1)^{-1} M_m 1, 1\right) \\
&= \sigma_{l,m}.
\end{aligned}$$

Hence the statement. ■

3 Visited points in high dimensions

In the high dimensional case we find that $E_d(n)$ grows fast, i.e. linearly in n, as we could have conjectured it from the transiency of the RWwIS. In Theorem 3 we prove this fact and compute remainder terms, too. Our approach is based on the one of [3], but there are some main differences. First, we have to consider the reversed random walk which is trivial in the case of [3]. After it, the renewal equation is written with matrices and vectors, which is more technical then in the case of [3]. Moreover, there will be a technical difficulty, namely we will have to consider the case, when the distribution of ε_0 is arbitrary. This will be treated separately in Proposition 4. After it, we will be able to estimate $V_d(n)$. In fact, $o(n^2)$ is enough for proving weak law of large numbers, and $O\left(n^{2-\delta}\right)$ for strong law of large numbers, but our estimations will be sharper. Nevertheless, these estimations are weaker then the ones of [3] because a symmetry argument, used in [3], fails here. That is why the computation is longer and it uses Proposition 4, too. Let us see the details.

Theorem 3 *Let $d \geq 3$. Assuming that ε_0 is distributed according to its unique stationary measure, we have*

$$\begin{aligned} E_3(n) &= n\gamma_3 + O(\sqrt{n}) \\ E_4(n) &= n\gamma_4 + O(\log n) \\ E_d(n) &= n\gamma_d + \beta_d + O(n^{2-d/2}) \qquad \textit{for } d \geq 5 \end{aligned}$$

with some constants γ_d, β_d, depending on the RWwIS.

Proof. Fix some dimensions $d \geq 3$. For the simplicity of notation we skip the index d in the sequel. Consider an $\{\xi_k = (\eta_k, \varepsilon_k), 0 \leq k\}$ RWwIS fulfilling our assumptions. Let

$$\left\{\widetilde{\xi}_k = (\widetilde{\eta}_k, \widetilde{\varepsilon}_k), 0 \leq k\right\}$$

be the reversed walk, i.e. for which the transition probabilities are defined by (9). Put $\eta_0 = 0$, $\gamma(0) = 1$ and define

$$\gamma(n) = P\left(\eta_n \notin \{\eta_0, ..., \eta_{n-1}\}\right)$$

which is just the probability that the walk visits a new point at step n. Obviously

$$\begin{aligned}\gamma(n) &= P(\eta_i \neq \eta_n \quad i = 0, ..n-1)\\ &= P(\eta_n - \eta_i \neq 0 \quad i = 0, ..n-1)\\ &= P(\widetilde{\eta}_{n-i} \neq 0 \quad i = 0, ..n-1)\\ &= P(\widetilde{\eta}_j \neq 0 \quad j = 1, ..n).\end{aligned}$$

It is clear that we have to examine the reversed walk.

Define $U_k \in \mathbb{R}^{s\times s}$ with

$$(U_k)_{i,j} = P\left(\widetilde{\xi}_k = (0,j)|\widetilde{\xi}_0 = (0,i)\right)$$

and $R_k \in \mathbb{R}^s$ with

$$(R_k)_j = P(0 \notin \{\widetilde{\eta}_1, ..., \widetilde{\eta}_k\}\, |\widetilde{\xi}_0 = (0,j)).$$

Put $\underline{1} \in \mathbb{R}^s$, $\underline{1} = (1,1,...,1)^T$. Obviously, we have:

$$\sum_{k=0}^{n} U_k \cdot R_{n-k} = \underline{1}.$$

We are interested in $\langle R_n, \mu\rangle = \gamma(n)$. From the definition of R_k, for $n_1 > n_2$ we have $R_{n_2} - R_{n_1} \geq \underline{0}$, which means that all the components of the vector are non-negative.

We know from Proposition 3 and [8] Theorem 5.2. that $(U_k)_{i,j} = c_j k^{-\frac{d}{2}} + o_{i,j}(k^{-\frac{d}{2}})$. Here we have $c_j = c\mu_j$, but this fact will not be used. So we have

$$\left(\sum_{k=0}^{n} U_k\right)_{i,j} = \widetilde{c}_{i,j} + O\left(n^{1-\frac{d}{2}}\right).$$

Using the monotonity of R_k we infer

$$\underline{1} \geq \left(\sum_{k=0}^{n} U_k\right) \cdot R_n.$$

Defining $\widehat{c}_j$ the following way

$$\left(\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{k=0}^{n} U_k\right)\right)_j = \frac{1}{s}\sum_{i=1}^{s}\left(\widetilde{c}_{i,j} + O\left(n^{1-\frac{d}{2}}\right)\right) = \widehat{c}_j + O\left(n^{1-\frac{d}{2}}\right),$$

we have

$$1 \geq \left\langle \left(\widehat{c}_1 + O\left(n^{1-\frac{d}{2}}\right), ..., \widehat{c}_s + O\left(n^{1-\frac{d}{2}}\right)\right), R_n \right\rangle. \tag{10}$$

For all j, $(R_n)_j$ has limit in n being a decreasing non-negative sequence. So let $(R_n)_j = R^j + a_n^j$, where $a_n^j \searrow 0$. It will be enough to estimate the order of a_n^j, because $\gamma(n) = \sum_{j=1}^{s} \mu_j \left(R^j + a_n^j\right)$.

For the estimation of the other direction let $k < n$. We have:

$$\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=0}^{k} U_i\right) \cdot R_{n-k} + \left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=k+1}^{n} U_i\right) \cdot \underline{1} \geq 1.$$

Since $(U_k)_{i,j} \geq 0$ for all k, i, j, we have $\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=0}^{k} U_i\right) \leq (\widehat{c}_1, ..., \widehat{c}_s)$. On the other hand, $\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=k+1}^{n} U_i\right) \cdot \underline{1} = o(1)$, as $k \to \infty$, thus

$$\langle (\widehat{c}_1, ..., \widehat{c}_s), R_{n-k} \rangle \geq 1 + o(1). \tag{11}$$

So if we let $n \to \infty$, $k \to \infty$, $n - k \to \infty$, (11) together with (10) yields

$$\widehat{c}_1 R^1 + ... + \widehat{c}_s R^s = 1.$$

Substituting to (10) we have:

$$\sum_{j=1}^{s} \left[\left(\widehat{c}_j + O\left(n^{1-\frac{d}{2}}\right)\right)\left(R^j + a_n^j\right)\right] \leq 1,$$

so

$$\sum_{j=1}^{s} \left[\widehat{c}_j a_n^j + O\left(n^{1-\frac{d}{2}}\right) R^j + O\left(n^{1-\frac{d}{2}}\right) a_n^j\right] \leq 0,$$

whence

$$\sum_{j=1}^{s} \widehat{c}_j a_n^j \leq O\left(n^{1-\frac{d}{2}}\right).$$

Since $\widehat{c}_j > 0$ and $a_n^j \geq 0$, we get that $a_n^j = O\left(n^{1-\frac{d}{2}}\right)$ for $1 \leq j \leq s$. This yields $\gamma(n) = \sum_{j=1}^{s} \mu_j \left(R^j + a_n^j\right) = \gamma + O\left(n^{1-\frac{d}{2}}\right)$. Hence the statement (just like in [3]). ■

Proposition 4 *The assertion of Theorem 3 remains true when the distribution of ε_0 is arbitrary.*

Proof. With the notation $\gamma(n) = \gamma + h(n)$ we already know that $h(n) = O\left(n^{1-\frac{d}{2}}\right)$. Define $\widetilde{\gamma}^{e_j}(n) = P\left(\eta_n \notin \{\eta_0, ..., \eta_{n-1}\} | \varepsilon_0 = j\right)$ and $\widetilde{\gamma}^{e_j}(n) = \gamma + h^j(n)$ for $j = 1, ..., s$. As in the previous proof, it would be sufficient to prove $h^j(n) = O\left(n^{1-\frac{d}{2}}\right)$ for all j.

For the present, let K be a fixed, great natural number, and

$$\mu_k + b_k^j(K) = P(\varepsilon_K = k \mid \varepsilon_0 = j) \quad j,k = 1,...s.$$

We know from the ergodic theorem of Markov chains that $b_k^j(K)$ tends to zero exponentially fast in K.

Denote by $p(K,n)$ the probability of visiting such a site at time n that was visited during the first K steps, but was not visited in the following $(n-K-1)$ steps provided that $\varepsilon_0 = j$. We know from [8] Theorem 5.2. that $p(K,n) = O\left(K \cdot (n-K)^{-\frac{d}{2}}\right)$, whence

$$\widetilde{\gamma}^{e_j}(n) = \sum_{k=1}^{s} \left[\left(\mu_k + b_k^j(K)\right) \widetilde{\gamma}^{e_k}(n-K)\right] + O\left(K \cdot (n-K)^{-\frac{d}{2}}\right). \tag{12}$$

Recall $\widetilde{\gamma}^{e_j}(n) = \gamma + h^j(n)$ to get

$$h^j(n) = \sum_{k=1}^{s} \mu_k h^k(n-K) + \sum_{k=1}^{s} b_k^j(K) h^k(n-K) + O\left(K \cdot (n-K)^{-\frac{d}{2}}\right) =: I + II + III. \tag{13}$$

Now let us set $K = K(n) = \lfloor n^\alpha \rfloor$, with arbitrary $0 < \alpha < 1$. It is clear that I is equal to $h(n-K)$, so since Theorem 3, $I = O\left((n-n^\alpha)^{1-\frac{d}{2}}\right) \leq O\left(n^{1-\frac{d}{2}}\right)$. Since $b_k^j(K)$ tends to zero exponentially fast in K, we have $II \leq O\left(n^{1-\frac{d}{2}}\right)$. Finally $III = O\left(n^\alpha (n-n^\alpha)^{-\frac{d}{2}}\right) \leq O\left(n^{1-\frac{d}{2}}\right)$. Hence the statement. ■

Now let us see the estimation of $V_d(n)$.

Theorem 4 *For $d \geq 3$ assuming that $\varepsilon_0 \sim \mu$ we have*

$$V_d(n) = O\left(n^{1+\frac{2}{d}}\right).$$

Proof. Let $\gamma_d(n,m)$ denote the probability of the event that our RWwIS visits new points in both the n^{th} and the m^{th} step under the condition that $\varepsilon_0 \sim \mu$, and let $A = \{S_d(i) \neq S_d(m), \quad i = 0,...,m-1\}$ (where, of course $S_d(i) = \sum_{j=1}^{i} X_j$). Obviously $\gamma_d(n,m) = \gamma_d(m,n)$, so w.l.o.g. when estimating $\gamma_d(n,m)$ one can assume $n > m$.

$$\begin{aligned}
\gamma_d(m,n) &= P(A \;\&\; S_d(j) \neq S_d(n), \quad j = 0,...,n-1) \\
&\leq P(A \;\&\; S_d(j) \neq S_d(n), \quad j = m,...,n-1) \\
&= \gamma(n) P(S_d(j) \neq S_d(n), \quad j = m,...,n-1 \quad | \quad A).
\end{aligned}$$

Here $P\left(S_d(j)\neq S_d(n),\quad i=m,...,n-1\quad|\quad A\right)$ is the probability of the event that the RWwIS visits a new point in the $(n-m)^{th}$ step, assuming that $\varepsilon_0\sim\mu\left(n\right)$. So the condition A is involved in $\mu\left(n\right)$, and because of the Markov property, it has no other contribution. This event is denoted by $\widetilde{\gamma}_d^{\mu(n)}(n-m)$. Because of Proposition 4 we know that $\widetilde{\gamma}_d^{\mu(n)}(n-m)\to\gamma_d$, as $(n-m)\to\infty$, and it is easy to see that this convergence is uniform in $\mu\left(n\right)$. So we know that for $\forall\delta>0\ \exists\ N=N\left(\delta\right)$, such that for $\forall n-m>N$ the following estimation holds.

$$\widetilde{\gamma}_d^{\mu(n)}(n-m)=\sum_{j=1}^{s}\mu(n)_j\widetilde{\gamma}_d^{e_j}(n-m)<(1+\delta)\gamma_d(n-m).$$

In addition, using Proposition 4 one can estimate $N\left(\delta\right)$, which will be done a little bit later. Now, let us see the estimation of $V_d\left(n\right)$

$$\begin{aligned}
V_d\left(n\right) &= \sum_{i,j=0}^{n}\gamma_d\left(i,j\right)-\sum_{i=0}^{n}\gamma_d\left(i\right)\sum_{j=0}^{n}\gamma_d\left(j\right)\\
&\leq 2\sum_{0\leq i\leq j\leq n}\left(\gamma_d\left(i,j\right)-\gamma_d\left(i\right)\gamma_d\left(j\right)\right)\\
&\leq 2\sum_{0\leq i<i+K\leq j\leq n}\left(\gamma_d\left(i,j\right)-\gamma_d\left(i\right)\gamma_d\left(j\right)\right)+2\sum_{\substack{0\leq i\leq n-K\\ i\leq j\leq i+K}}\gamma_d\left(i,j\right)\\
&=:S_1+S_2.
\end{aligned}$$

Let K be big enough, such that for $n-m>K$ one would have $\widetilde{\gamma}_d(n-m)<(1+\delta)\gamma_d(n-m)$. Estimating S_1 and S_2 separately, we get

$$\begin{aligned}
\frac{S_1}{2} &= \sum_{i=0}^{n-K}\sum_{j=i+K}^{n}\gamma_d\left(i,j\right)-\sum_{i=0}^{n-K}\sum_{j=i}^{n}\gamma_d\left(i\right)\gamma_d\left(j\right)+\sum_{i=0}^{n-K}\sum_{j=i}^{i+K}\gamma_d\left(i\right)\gamma_d\left(j\right)\\
&\leq \sum_{i=0}^{n-K}\gamma_d\left(i\right)\max_{0\leq i\leq n-K}\left(\sum_{j=i}^{n}\left(1+\delta\right)\gamma_d\left(j-i\right)-\sum_{j=i}^{n}\gamma_d\left(j\right)\right)\\
&\quad+\sum_{i=0}^{n-K}\gamma_d\left(i\right)\sum_{j=i}^{i+K}\gamma_d\left(j\right)\\
&\leq \sum_{i=0}^{n-K}\gamma_d\left(i\right)\left[\delta E_d(n)+E_d(n-\left\lfloor\frac{n}{2}\right\rfloor)-E_d(n)+E_d(\left\lfloor\frac{n}{2}\right\rfloor)\right]\\
&\quad+\sum_{i=0}^{n-K}\gamma_d\left(i\right)K.
\end{aligned}$$

where estimating the maximum we used the monotonity of $\gamma_d(n)$, too. On the other hand

$$S_2 \leq 2 \sum_{\substack{0\leq i\leq n-K \\ i\leq j\leq i+K}} \gamma_d\left(i\right) \leq 2KE_d(n).$$

From Proposition 4 one can easily deduce that $\widetilde{\gamma}_d^{\nu}(k) < \left(1+O(k^{1-\frac{d}{2}})\right)\gamma_d(k)$, uniformly in ν. So replacing K to $K\left(n\right)$ in the above argument, one can change δ to $O\left(K(n)^{1-\frac{d}{2}}\right)$, thus

$$\begin{aligned}
V_3\left(n\right) &\leq O(n)\left[O\left(K(n)^{1-\frac{d}{2}}\right)O\left(n\right)+O\left(\sqrt{n}\right)\right]+K\left(n\right)O\left(n\right) \\
V_4\left(n\right) &\leq O(n)\left[O\left(K(n)^{1-\frac{d}{2}}\right)O\left(n\right)+O\left(\log n\right)\right]+K\left(n\right)O\left(n\right) \\
V_d\left(n\right) &\leq O(n)\left[O\left(K(n)^{1-\frac{d}{2}}\right)O\left(n\right)+O\left(1\right)\right]+K\left(n\right)O\left(n\right) \quad d\geq 5.
\end{aligned}$$

Now $K(n) = n^{\frac{2}{d}}$ proves the statement. ■

Corollary 1 *For RWwIS in $d \geq 3$ the weak law of large numbers holds, namely*

$$P\left(\left|L_d\left(n\right)-E_d\left(n\right)\right| > \varepsilon E_d\left(n\right)\right) \to 0$$

for $\forall\varepsilon > 0$.

Proof. Since $V_d(n) = o\left(n^2\right)$ Chebyshev's inequality applies (just like in [3]). ■

From Theorem 4 one can deduce even strong law of large numbers:

Theorem 5 *For RWwIS in $d \geq 3$ strong law of large numbers holds, namely*

$$P\left(\lim_{n\to\infty}\frac{L_d(n)}{E_d(n)}=1\right)=1.$$

Proof. The proof is almost the same as in [3].

Let α be any real number satisfying

$$\frac{8}{9} < \alpha < 1 \tag{14}$$

and take for β any number with

$$\frac{1}{2\alpha-5/3} < \beta < \frac{1}{1-\alpha} \tag{15}$$

such a choice of β is possible because of (14).Put

$$n_k = \lfloor k^\beta \rfloor \quad k = 1, 2, \ldots . \tag{16}$$

Since Theorem 4 we have $V_d(n) = O\left(n^{5/3}\right)$. Now apply Chebyshev's inequality to have

$$P\left(|L_d(n_k) - n_k\gamma_d| > n_k^\alpha\right) = O\left(n_k^{5/3-2\alpha}\right) = O\left(k^{(10/3-4\alpha)\beta/2}\right).$$

Since $(10/3 - 4\alpha)\beta/2 < -1$ by (15) it follows that

$$\sum_{k=1}^{\infty} P\left(|L_d(n_k) - n_k\gamma_d| > n_k^\alpha\right) < \infty.$$

Hence, by the Borel-Cantelli lemma, there is probability 1 that

$$|L_d(n_k) - n_k\gamma_d| \le n_k^\alpha \tag{17}$$

hold for all sufficiently large k. But (17) implies

$$|L_d(n) - n\gamma_d| \le |L_d(n) - L_d(n_k)| + |L_d(n_k) - n_k\gamma_d| + |(n_k - n)\gamma_d| \le n_k^\alpha + 2n_{k+1} - 2n_k \tag{18}$$

for $n_k \le n < n_{k+1}$. By (16), $n_{k+1} - n_k = O\left(k^{\beta-1}\right)$, i.e.

$$\lim_{k\to\infty} \frac{(k+1)^\beta - k^\beta}{k^{\beta-1}} = \beta.$$

Since $\beta - 1 < \alpha\beta$ we have also $k^{\beta-1} = O\left(n_k^\alpha\right)$. Thus the right side of (18) is $O\left(n_k^\alpha\right)$ and hence $O\left(n^\alpha\right)$ for $n_k \le n < n_{k+1}$. Thus we have proved that for almost all paths

$$L_d(n) = n\gamma_d + O\left(n^\alpha\right)$$

for every $\alpha > 8/9$. Hence the statement. ■

It is easy to check that if we had estimation $V_d(n) = O\left(n^\tau\right)$ with some $\tau < 2$, then the above argument would work with the following parameters:

$$\begin{aligned} \frac{1+\tau}{3} &< \alpha < 1 \\ \frac{1}{2\alpha - \tau} &< \beta < \frac{1}{1-\alpha} \end{aligned}$$

So the main point is that we should have some $\tau < 2$ such that $V_d(n) = O\left(n^\tau\right)$ as it was mentioned at the beginning of the Section.

Identifying the constant γ_d is an interesting question, though we can give a closed form only in the case of SSRW.

Remark 2 *For the constant* γ_d *we have*

$$\gamma_d = P(\eta_k \neq 0 : k \geq 1 | \varepsilon_0 \sim \mu)$$

Proof. First of all, we prove that the constant γ_d is the same for both the primary and the reversed walk. As we have weak law of large numbers if we had $\gamma_d \neq \widetilde{\gamma}_d$ for the primary and the reversed walk then for large enough n the following would hold

$$P\left(|L_d(n) - n\gamma_d| < \varepsilon n\gamma_d \text{ and } \left|\widetilde{L}_d(n) - n\widetilde{\gamma}_d\right| < \varepsilon n\widetilde{\gamma}_d\right) > 1 - \varepsilon,$$

where $L_d(n)$ and $\widetilde{L}_d(n)$ are identically distributed random variables, as the number of visited points does not depend on the order of counting them. The choice $\varepsilon = \frac{1}{2}|\gamma_d - \widetilde{\gamma}_d|$ leeds to contradiction.

We have seen that

$$\gamma_d(n) = P(Y_1 + Y_2 ... + Y_j \neq 0 \quad j = 1, ..n).$$

Taking $n \to \infty$ the assertion follows. ■

The same is true in the case of SSRW, namely γ_d is the probability that the random walk never returns to the origin. Here, $\gamma_d = \frac{1}{U_d}$ with $U_d = \sum_{k=1}^{s} P(X_1 + X_2 ... + X_j \neq 0)$. It is proved that

$$U_d = d \int_0^\infty \exp(-dt) I_0(t)^d dt,$$

where

$$I_0(t) = \frac{1}{\pi} \int_0^\pi \exp(t \cos\theta) d\theta$$

is the so-called modified Bessel function of first kind (see, for instance, [7], page 370).

4 Visited points in two dimensions

In this section we calculate $E_2(n)$ and estimate $V_2(n)$.

4.1 Simulations of $E_2(n)$

Before proving Theorem 6 I made some simulations to conjecture, whether $E_2(n)$ is of order $\frac{n}{\log n}$ or not. In fact, by that time I had no idea how to prove the theorem in a rigorous manner.

Let us consider three random walks: B_1, B_2, B_3. B_1 is the simple symmetric random walk. B_2 is a very simple RWwIS: the Q matrix corresponding to B_2 is $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, and for the four possible transitions of internal states B_2 steps with the four unit vectors, respectively, with probability 1. Assuming that $\varepsilon_0 \sim (1/2, 1/2)$, it is clear that the steps of B_2 are identically distributed to the ones of B_1, but they are dependent. Define B_3 with the appropriate probabilities

$$\begin{aligned} p_{(0,1),1,1} &= \frac{3}{10}, \\ p_{(1,0),1,2} &= \frac{7}{10}, \\ p_{(0,-1),2,1} &= \frac{5}{14}, \\ p_{(0,1),2,1} &= \frac{1}{7}, \\ p_{(-1,0),2,2} &= \frac{1}{2}. \end{aligned}$$

It is easy to see that B_1, B_2 and B_3 fulfill the essential basic assumptions, i.e. assumption (i) (iii) and (iv).

For these three random walks I generated approximately 10^3 trajectories of 10^4 steps, 10^2 trajectories of 10^5 steps, 10 trajectories of 10^6 steps and 1 trajectory of 10^7 steps using Mathematica. For each trajectory I computed the number of distinct sites visited by the random walk. After it, from each sample I computed the mean, and assuming that this value is $\frac{c_i \log n}{n}$ with some constant c_i ($i = 1, 2, 3$, where c_i corresponds to B_i) I got estimations to the c_i's. The result of these estimations can be seen in the following table where $\widehat{c_i}(n)$ denotes the estimation of c_i from trajectories of length n.

n	$\widehat{c}_1(n)$	$\widehat{c}_2(n)$	$\widehat{c}_3(n)$
10^4	2.65987	2.67432	2.71659
10^5	2.73242	2.79778	2.79516
10^6	2.84455	2.77546	2.79338
10^7	2.8126	2.67285	3.00839

Since $c_1 = \pi$ (see Theorem 1 in [3]) apparently these step sizes are not sufficient to conjecture the size of the constant, but rather to see the order of increase. Since the behavior in the last two cases are very similar to the one of the simple symmetric random walk, these results suggested the desired asymptotic behavior and inspired me to make the calculations of the proofs of Theorem 2 and Theorem 6. Although Theorem 6 does not concern the above B_2 and B_3 as they do not fulfill our basic assumption (ii), these results are inspirational because basic assumption (ii) is not essential (see Final remark 2).

4.2 Analytical arguments

In this subsection, we will see the formal proofs of the estimations of $E_2(n)$ and $V_2(n)$. The idea of the proofs (assuming that $\varepsilon_0 \sim \mu$) is similar to the ones of Theorem 3 and 4, or [3] Theorem 1 and Theorem 2. The computations are longer than in [3]. We have to write the renewal equation in terms of vectors and matrices, which is a new idea, and we use the above proved Proposition 1 because it is essential that the remainder term of the probability of returning to the origin should be summable, which was trivial in the case of [3]. We have to consider the case of arbitrary initial distribution, separately, just like in Section 3. In this case, we formulate the fact that after some steps the distribution of ε will be very close to μ.

Theorem 6 *Let $d = 2$. Assuming that $\varepsilon_0 \sim \mu$ and (1) exists, we have*

$$E_2(n) = \frac{2\pi\sqrt{|\sigma|}n}{\log n} + O\left(\frac{n \log\log n}{\log^2 n}\right).$$

Proof. As in the proof of Theorem 3, we examine the reversed RWwIS and write the renewal equation

$$\sum_{k=0}^{n} U_k \cdot R_{n-k} = \underline{1}. \tag{19}$$

Proposition 1 yields

$$(U_k)_{i,j} = \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \frac{1}{k} + O\left(k^{-3/2}\right),$$

thus

$$\left(\sum_{k=0}^{n} U_k\right)_{i,j} = \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log\left(c_{i,j} n\right) + O\left(n^{-1/2}\right). \tag{20}$$

Our purpose is to estimate $\langle R_n, \mu \rangle = \gamma(n)$. Exactly as in the high dimensional case, R_n is decreasing, so (19) yields

$$\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=0}^{k} U_l\right) \cdot R_{n-k} + \left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=k+1}^{n} U_l\right) \cdot \underline{1} \geq 1. \tag{21}$$

Let $k \to \infty$, $n \to \infty$. The relation between k and n will be fixed later. From (20) it follows that

$$\left[\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=0}^{k} U_l\right)\right]_j = \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log\left(\widehat{c}_j k\right) + O\left(k^{-1/2}\right) \tag{22}$$

for some $\widehat{c}_j$. So we have for $k < n$

$$\begin{aligned}\left[\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=k+1}^{n} U_l\right)\right]_j &= \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \left[\log\left(\widehat{c}_j n\right) - \log\left(\widehat{c}_j k\right)\right] + O\left(k^{-1/2}\right) \qquad (23)\\ &= \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log\frac{n}{k} + O\left(k^{-1/2}\right).\end{aligned}$$

Substituting (22) and (23) to the left hand side of (21) we get

$$\sum_{j=1}^{s}\left[\frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log\left(\widehat{c}_j k\right) + O\left(k^{-1/2}\right)\right]\left(R_{n-k}\right)_j + \sum_{j=1}^{s}\frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log\frac{n}{k} + O\left(k^{-1/2}\right). \tag{24}$$

Put $k = \left\lfloor n - \frac{n}{\log n}\right\rfloor$. This yields $\log k \sim \log(n-k)$. Using $\gamma(n-k) = \sum_{j=1}^{s}\mu_j \left(R_{n-k}\right)_j$, (24) can be written as

$$\begin{aligned}&\gamma(n-k)\left[\frac{1}{2\pi\sqrt{|\sigma|}}\log k\right] + \sum_{j=1}^{s}\left[\frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log\widehat{c}_j + O\left(k^{-1/2}\right)\right]\left(R_{n-k}\right)_j \qquad (25)\\ &+C\log\frac{n}{k} + O\left(k^{-1/2}\right).\end{aligned}$$

Since $\log\frac{n}{k} \to 0$, and $\left(R_{n-k}\right)_j \to 0$, as $n-k \to \infty$ (the latter is the recurrence property of the two dimensional RWwIS, which is proved in [15]), it follows that

$$\gamma(n-k) \geq \frac{2\pi\sqrt{|\sigma|}}{\log k} + o\left(\frac{1}{\log k}\right). \tag{26}$$

Hence, by the choice of k,

$$\gamma(n-k) \geq \frac{2\pi\sqrt{|\sigma|}}{\log(n-k)} + o\left(\frac{1}{\log(n-k)}\right). \tag{27}$$

Now let us give an upper estimation to $\gamma(n)$. From (19) it follows that

$$\left(\sum_{k=0}^{n} U_k\right) \cdot R_n \leq \underline{1}.$$

Multiplying by the vector $\frac{1}{s}\underline{1}$ we get

$$\sum_{j=1}^{s}\left[\frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log(\widehat{c}_j n) + O\left(n^{-1/2}\right)\right](R_n)_j \leq 1,$$

thus

$$\begin{aligned} & S_1 + S_2 + S_3 \\ : \quad & = \frac{1}{2\pi\sqrt{|\sigma|}}\sum_{j=1}^{s}\mu_j (R_n)_j \log n + \frac{1}{2\pi\sqrt{|\sigma|}}\sum_{j=1}^{s}\mu_j (R_n)_j \log\widehat{c}_j + \sum_{j=1}^{s} O\left(n^{-1/2}\right)(R_n)_j \leq 1. \end{aligned}$$

Since $(R_n)_j \to 0$, it follows that $S_2 + S_3 = o(1)$. So we have the upper estimation

$$\gamma(n) \leq \frac{2\pi\sqrt{|\sigma|}}{\log n} + o\left(\frac{1}{\log n}\right). \tag{28}$$

From (27) and (28) we get

$$\gamma(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + o\left(\frac{1}{\log n}\right). \tag{29}$$

Unfortunately, the estimation (29) is not good enough for our purposes. But (28) yields $(R_n)_j = O\left(\frac{1}{\log n}\right)$ for all $1 \leq j \leq s$. Hence, with the previous notation, $S_2 = O\left(\frac{1}{\log n}\right)$. Obviously $S_3 = O\left(\frac{1}{\log n}\right)$. Thus we arrived at

$$\frac{1}{2\pi\sqrt{|\sigma|}}\sum_{j=1}^{s}\mu_j (R_n)_j \log n \leq 1 + O\left(\frac{1}{\log n}\right).$$

Hence

$$\gamma(n) \leq \frac{2\pi\sqrt{|\sigma|}}{\log n} + O\left(\frac{1}{\log^2 n}\right). \tag{30}$$

This estimation is sharp enough, as we will see later.

Now we have to improve our lower estimation. From (29) and (25) it follows that there exist C_1 and C_2 constants, such that

$$\gamma(n-k)\left[\frac{1}{2\pi\sqrt{|\sigma|}}\log k+\frac{C_2}{\log(n-k)\gamma(n-k)}\right]+C_1\log\frac{n}{k}+O\left(k^{-\frac{1}{2}}\right)\geq 1.$$

Using $\gamma(n-k)\log(n-k)\geq 2\pi\sqrt{|\sigma|}+o(1)$, we conclude

$$\gamma(n-k)\left[\frac{1}{2\pi\sqrt{|\sigma|}}\log k+O(1)\right]+C\log\frac{n}{k}+O\left(k^{-\frac{1}{2}}\right)\geq 1,$$

thus

$$\gamma(n-k)\log(n-k)\geq\left(2\pi\sqrt{|\sigma|}-C2\pi\sqrt{|\sigma|}\log\frac{n}{k}+O\left(k^{-\frac{1}{2}}\right)\right)\frac{\log(n-k)}{\log k+O(1)}.$$

From now the end of the proof is almost the same as is [3]. We claim that

$$\gamma(l)\log l\geq 2\pi\sqrt{|\sigma|}+O\left(\frac{\log\log l}{\log l}\right). \tag{31}$$

From (30) and (31) the statement would follow just like in [3]. Put $l=\frac{n}{\log n}$. It is obvious that

$$\frac{\log\log l}{\log l}\sim\frac{\log\log n}{\log n}. \tag{32}$$

To prove (31), observe that

$$\lim_{n\to\infty}\log\left(\frac{1}{1-\frac{1}{\log n}}\right)\frac{\log n}{\log\log n}=0. \tag{33}$$

In the sense of (32) and (33) in order to prove (31) it is enough to verify

$$\frac{\log\left(\frac{n}{\log n}\right)}{\log\left(n\left(1-\frac{1}{\log n}\right)\right)+O(1)}=1+O\left(\frac{\log\log l}{\log l}\right). \tag{34}$$

This is just an elementary computation.

$$\begin{aligned}\frac{\log n-\log\log n}{\log n+\log\left(1-\frac{1}{\log n}\right)+O(1)} &= \frac{1-\frac{\log\log n}{\log n}}{1-\frac{\log\left(1-\frac{1}{\log n}\right)}{\log n}+\frac{O(1)}{\log n}}\\ &= \frac{1-\frac{\log\log n}{\log n}}{1+\frac{O(1)}{\log n}}\\ &= \left(1-\frac{\log\log n}{\log n}\right)\left(1+\frac{O(1)}{\log n}\right)\\ &= 1+O\left(\frac{\log\log n}{\log n}\right).\end{aligned}$$

So we have proved (31). (30) and (31) imply

$$\gamma(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + O\left(\frac{\log\log n}{\log^2 n}\right). \tag{35}$$

Now, an elementary calculation completes the proof. Obviously it is enough to prove

$$\sum_{i=3}^{n}\left(\frac{1}{\log i} + O\left(\frac{\log\log i}{\log^2 i}\right)\right) = \frac{n-2}{\log n} + O\left(\frac{n\log\log n}{\log^2 n}\right). \tag{36}$$

It is trivial that $\sum\limits_{i=3}^{n}\frac{1}{\log i} \geq \frac{n-2}{\log n}$. First we are going to prove that

$$\sum_{i=3}^{n}\frac{1}{\log i} \leq \frac{n-2}{\log n} + O\left(\frac{n\log\log n}{\log^2 n}\right). \tag{37}$$

Obviously we have

$$\sum_{i=3}^{n}\frac{1}{\log i} = \frac{n-2}{\log n} + \sum_{i=3}^{n}\left(\frac{1}{\log i} - \frac{1}{\log n}\right) = \frac{n-2}{\log n} + \frac{1}{\log n}\sum_{i=3}^{n}\frac{\log(n/i)}{\log i}.$$

So in order to show (37) it is enough to verify

$$\frac{\log n}{n\log\log n}\sum_{i=3}^{n}\frac{\log(n/i)}{\log i} < C < \infty \tag{38}$$

for all n. Denote $i_0 = i_0(n) = \left\lfloor \frac{n\log\log n}{\log^2 n}\right\rfloor$. Now we have

$$S_1(n) + S_2(n) = \frac{\log n}{n\log\log n}\sum_{i=3}^{i_0}\frac{\log(n/i)}{\log i} + \frac{\log n}{n\log\log n}\sum_{i=i_0}^{n}\frac{\log(n/i)}{\log i}.$$

Thus

$$S_1(n) \leq \frac{1}{n\log\log n}\sum_{i=3}^{i_0}\frac{\log^2 n}{\log i} \leq \frac{1}{\log 3}$$

because of the definition if i_0. On the other hand for all $\varepsilon > 0$

$$S_2(n) \leq \frac{1}{\log\log n}\frac{\log n}{\log i_0}\log\left(\frac{n}{i_0}\right) \leq C\frac{\log\log^2 n - \log\log\log n + \varepsilon}{\log\log n} \leq 2C$$

holds for large enough n where C is an upper bound of $\frac{\log n}{\log i_0}$ (such an upper bound exists as $\frac{\log n}{\log i_0} \to 1$). Hence we have proved (38).

Now we are going to prove that

$$S(n) = \frac{\log^2 n}{n \log\log n} \sum_{i=3}^{n} O\left(\frac{\log\log i}{\log^2 i}\right) \tag{39}$$

is a bounded series. There exists a bounded a_n series such that $S(n)$ can be written as

$$\frac{\log^2 n}{n \log\log n} \sum_{i=3}^{n} a_i \frac{\log\log i}{\log^2 i}.$$

Now we use the same trick as previously, namely we cut the sum into two pieces. Denote $i_1 = \left\lfloor \frac{n}{\log^2 n} \right\rfloor$, $c = \max_i a_i$, and write

$$S(n) \leq S_1'(n) + S_1'(n) = c \sum_{i=3}^{i_1} \frac{\log^2 n}{n \log^2 i} + c \sum_{i=i_1}^{n} \frac{\log^2 n}{n \log^2 i}.$$

Now we have

$$S_1'(n) \leq c i_1 \frac{\log^2 n}{n} \leq c$$

and for all $\varepsilon > 0$

$$S_2'(n) \leq cn \frac{1}{n} \frac{\log^2 n}{\log^2 i_1} \leq c \frac{\log^2 n}{\log^2 n - 4 \log\log n - \varepsilon} \leq 2c$$

for all sufficiently large n. So we have proved (39). Hence the theorem. ■

As in the high dimensional case, the initial distribution does not influence the asymptotic behavior. More precisely

Proposition 5 *The assertion of Theorem 6 remains true when the distribution of* ε_0 *is arbitrary.*

Proof. The proof is very similar to the one of Proposition 4. We know that

$$\gamma(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + O\left(\frac{\log\log n}{\log^2 n}\right).$$

With the notation $\widetilde{\gamma}^{e_j}(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + h^j(n)$ our aim is to prove $h^j(n) = O\left(\frac{\log\log n}{\log^2 n}\right)$. The analogue of (12) is

$$\frac{2\pi\sqrt{|\sigma|}}{\log n} + h^j(n) = \sum_{k=1}^{s} \left[\left(\mu_k + b_k^j(K)\right) \left(\frac{2\pi\sqrt{|\sigma|}}{\log(n-K)} + h^k(n-K) \right) \right] + O\left(K \cdot (n-K)^{-1}\right),$$

and the analogue of (13) is

$$\begin{aligned} h^j(n) &= \sum_{k=1}^{s}\mu_k h^k(n-K) + \sum_{k=1}^{s} b_k^j(K) h^k(n-K) + O\left(K\cdot(n-K)^{-1}\right) \\ &\quad + \left(\frac{2\pi\sqrt{|\sigma|}}{\log(n-K)} - \frac{2\pi\sqrt{|\sigma|}}{\log n}\right) \\ &=: I + II + III + IV. \end{aligned}$$

With the choice $K(n) = \lfloor\sqrt{n}\rfloor$ elementary calculations show that $I + II + III + IV \leq O\left(\frac{\log\log n}{\log^2 n}\right)$. ■

Now let us see the estimation of the variance.

Theorem 7 *If $\varepsilon_0 \sim \mu$ then we have*

$$V_2(n) = O\left(\frac{n^2 \log\log n}{\log^3 n}\right).$$

Proof. The beginning of the proof is the same as in Theorem 4. The difference is that when we change K to $K(n)$, we can write $O\left(\frac{\log\log K(n)}{\log K(n)}\right)$ instead of δ in the sense of Proposition 5. From now, just like in the proof of Theorem 4 it is not difficult to deduce

$$V_2(n) \leq O\left(\frac{n}{\log n}\right)\left[\frac{\log\log K(n)}{\log K(n)} O\left(\frac{n}{\log n}\right) + O\left(\frac{n\log\log n}{\log^2 n}\right)\right] + K(n)\, O\left(\frac{n}{\log n}\right).$$

Taking $K(n) = \left\lfloor \frac{n}{\log^2 n} \right\rfloor$ proves the statement. ■

Remark 3 *The assertion of Theorem 7 remains true when the distribution of ε_0 is some arbitrary ν. Moreover, the great order is uniform in ν.*

Proof. Let us introduce the notation $L_2^\nu(n)$, $E_2^\nu(n)$ and $V_2^\nu(n)$ for the RWwIS when $\varepsilon_0 \sim \nu$. Obviously,

$$V_2^\nu(n) = E\left[(L_2^\nu(n))^2\right] - (E_2^\nu(n))^2. \tag{40}$$

On the other hand,

$$\sum_{j=1}^{s} \nu_j V_2^{e_j}(n) = \sum_{j=1}^{s} \nu_j E\left[\left(L_2^{e_j}(n)\right)^2\right] - \sum_{j=1}^{s} \nu_j \left(E_2^{e_j}(n)\right)^2. \tag{41}$$

Since $E\left[(L_2^\nu(n))^2\right] = \sum_{j=1}^{s} \nu_j E\left[\left(L_2^{e_j}(n)\right)^2\right]$, subtracting (41) from (40) we conclude

$$V_2^\nu(n) - \sum_{j=1}^{s} \nu_j V_2^{e_j}(n) = O\left(\frac{n^2\log\log n}{\log^3 n}\right). \tag{42}$$

It is clear that the great order on the right hand side is uniform in ν. In the sense of (42) it is enough to prove the assertion for $\nu = e_j, (j = 1, ..., s)$. To do so, let us substitute $\mu = \nu$ to (42) to obtain

$$V_2(n) - \sum_{j=1}^{s} \mu_j V_2^{e_j}(n) = O\left(\frac{n^2 \log\log n}{\log^3 n}\right).$$

Now Theorem 7 implies

$$\sum_{j=1}^{s} \mu_j V_2^{e_j}(n) = O\left(\frac{n^2 \log\log n}{\log^3 n}\right).$$

Since for all j and n μ_j and $V_2^{e_j}(n)$ are non negative, we have proved the statement for all e_j. ■

Corollary 2 *For a RWwIS in $d = 2$ dimension weak law of large numbers holds.*

Proof. Since $O\left(\frac{n^2 \log\log n}{\log^3 n}\right) < O\left(\frac{n^2}{\log^2 n}\right)$, Chebyshev's inequality applies. ■

The proof of the strong law of large numbers is quite complicated, so we treat it in a different Section.

5 Law of large numbers in the plane

This whole Section is dedicated to the proof of the strong law in $d = 2$. The proof presented here is almost the repetition of the one in [3] with some elements of [11]. However, our case is less complicated than the one in [11], so the proof is shorter and more transparent.

Theorem 8 *For any RWwIS in $d = 2$ which fulfills our basic assumption and for which (6) exists, strong law of large numbers holds, namely*

$$P\left(\lim_{n\to\infty} \frac{L_2(n)}{E_2(n)} = 1\right) = 1.$$

First, we formulate a lemma, and show that using the lemma the statement of the above theorem follows.

Lemma 1 *There exist $\delta > 0$ such that for every $\varepsilon > 0$*

$$P\left(\left|L_2(n)-\frac{2\pi\sqrt{|\sigma|}n}{\log n}\right|>\varepsilon\frac{n}{\log n}\right)=O\left(\frac{1}{\log^{1+\delta} n}\right). \tag{43}$$

Once we have verified Lemma 1, we can argue the following way.

Let Θ be a number satisfying

$$\frac{1}{1+\delta}<\Theta<1, \tag{44}$$

and denote for $k\in\mathbb{N}$

$$n_k=\left\lfloor \exp\left(k^{\Theta}\right)\right\rfloor. \tag{45}$$

Now (43) and (44) yield

$$\sum_{k=0}^{\infty}P\left(\left|L_2(n_k)-\frac{2\pi\sqrt{|\sigma|}n_k}{\log n_k}\right|>\varepsilon\frac{n_k}{2\log n_k}\right)<\infty.$$

Thus we have from the Borel-Cantelli lemma that for almost all paths

$$\left|L_2(n_k)-\frac{2\pi\sqrt{|\sigma|}n_k}{\log n_k}\right|>\varepsilon\frac{n_k}{2\log n_k} \tag{46}$$

holds for sufficiently large k.

For $n_k\le n\le n_{k+1}$ we have

$$L_2(n_k)-\frac{2\pi\sqrt{|\sigma|}n_{k+1}}{\log n_{k+1}}<L_2(n)-\frac{2\pi\sqrt{|\sigma|}n}{\log n}<L_2(n_{k+1})-\frac{2\pi\sqrt{|\sigma|}n_k}{\log n_k} \tag{47}$$

Now (46) and the analogous formula with n_{k+1} yield that the absolute value of the extreme members of (47) is at most

$$\varepsilon\frac{n_{k+1}}{2\log n_{k+1}}+2\pi\sqrt{|\sigma|}\left(\frac{n_{k+1}}{2\log n_{k+1}}-\frac{n_k}{\log n_k}\right). \tag{48}$$

Since $\Theta<1$ it follows from (45) that, for large enough k, (48) is smaller than

$$\varepsilon\frac{n_k}{\log n_k}\le\varepsilon\frac{n}{\log n}. \tag{49}$$

Combining (46), (47) and (49) it follows that for every $\varepsilon>0$ and for almost all paths

$$\left|L_2(n)-\frac{2\pi\sqrt{|\sigma|}n}{\log n}\right|>\varepsilon\frac{n}{\log n}$$

for sufficiently large n. Th strong law of large number follows. So it remained to prove Lemma 1.

Proof. The proof of Lemma 1 falls naturally into two parts.

In the first part we estimate the probability of the event

$$L_2(n) > (1+\varepsilon)\frac{2\pi\sqrt{|\sigma|}n}{\log n}. \quad (50)$$

Let us introduce the notations

$$N = \left\lfloor \left(\frac{\log n}{\log\log n}\right)^{1/3} \right\rfloor, \quad (51)$$

and

$$n_i = \left\lfloor \frac{ni}{N} \right\rfloor$$

for $0 \le i \le N$. Furthermore, denote by A_i the event

$$|L_2(n_i) - L_2(n_{i-1})| > \left(1+\frac{\varepsilon}{2}\right)\frac{2\pi\sqrt{|\sigma|}n}{N\log n},$$

and by B_i the event

$$L_2(n_i) - L_2(n_{i-1}) > \frac{\varepsilon\pi\sqrt{|\sigma|}n}{\log n},$$

for $1 \le i \le N$. From Theorem 7, Remark 3 and Chebyshev's inequality we have

$$P(A_i) < C_1\frac{\log\log n}{\log n}, \quad (52)$$

where C_1 can depend on ε, but not on n, N, i (note that the variance estimate is uniform in the initial distribution). Moreover, we have

$$P(A_i) < C_2\frac{\log\log n}{N^2\log n}. \quad (53)$$

Observe that if at most one of the events $A_1, ..., A_N$ occurs and none of the events $B_1, ..., B_N$ occur, then the event (50) cannot occur either. Moreover, for the probability of A_i under the condition A_j $(j < i)$ the estimation (53) holds, as the condition has only a contribution via the initial distribution ε_{n_i}.So the probability of the event (50) can be estimated using (52) and (53) the following way

$$P\left(L_2(n) > (1+\varepsilon)\frac{2\pi\sqrt{|\sigma|}n}{\log n}\right) < C_1^2\left(\frac{N\log\log n}{\log n}\right)^2 + C_2\frac{\log\log n}{N\log n}.$$

Substituting (51) we obtain

$$P\left(L_2(n) > (1+\varepsilon)\frac{2\pi\sqrt{|\sigma|}n}{\log n}\right) < C_3\left(\frac{\log\log n}{\log n}\right)^{4/3}. \tag{54}$$

In the second part we estimate

$$P\left(L_2(n) < (1-\varepsilon)\frac{2\pi\sqrt{|\sigma|}n}{\log n}\right).$$

Denote

$$K = \lfloor \log\log n \rfloor,$$

and let M_{ij} $(1 \le i,j \le K)$ be the number of lattice points which are common in path parts M_i and M_j, where M_i denotes the set of points which are visited between $\lfloor (i-1)n/K \rfloor + 1$ and $\lfloor in/K \rfloor$ $(1 \le i \le K)$. Suppose for the moment that we can prove

$$\sup_{i<j} E\left(M_{ij}\right) = O\left(\frac{n\log\log n}{K\log^2 n}\right). \tag{55}$$

Then for every ϑ with $0 < \vartheta < 1$ we have

$$\sup_{i<j} P\left(M_{ij} > \frac{n\log\log n}{K\log^{1+\vartheta} n}\right) = O\left(\frac{1}{\log^{1-\vartheta} n}\right). \tag{56}$$

Let C_{ij} denote the event whose probability is estimated in (56). As (55) yields

$$\sup_{j} E\left(M_{1j}\right) = O\left(\frac{n\log\log n}{K\log^2 n}\right)$$

for arbitrary ν initial distribution of internal states, and under the condition C_{ij} the probability of $C_{i'j'}$ with $1 \le i < j < i' < j' \le K$ is only affected via the distribution of $\varepsilon_{i'}$, we conclude

$$\sup_{1\le i<j<i'<j'\le K} P(C_{i,j} \cap C_{i',j'}) = O\left(\frac{1}{\log^{2-2\vartheta} n}\right). \tag{57}$$

If we were able to prove

$$\sup_{i,j,i',j'} E\left(M_{ij}M_{i'j'}\right) = O\left(\frac{n^2\log^2\log n}{K^2\log^4 n}\right), \tag{58}$$

where the supremum is taken over indices for which $\#\{i,j,i',j'\} = 4$ and either $1 \le i < i' < j' < j \le K$ or $1 \le i < i' < j < j' \le K$ holds, then using

$$P\left(M_{ij} > \frac{n\log\log n}{K\log^{1+\vartheta} n}, M_{i'j'} > \frac{n\log\log n}{K\log^{1+\vartheta} n}\right) < P\left(M_{ij}M_{i'j'} > \frac{n^2\log^2\log n}{K^2\log^{2+2\vartheta} n}\right)$$

and (57) we could infer that the probability that two events C_{ij} and $C_{i'j'}$ with $\#\{i,j,i',j'\} = 4$ occur is

$$O\left(\frac{K^4}{\log^{2-2\vartheta} n}\right). \tag{59}$$

Now, for $i = 1, ..., K$ let us introduce the notation

$$D_i = \left\{ M_i \text{ covers less than } \left(1 - \frac{\varepsilon}{2}\right) \frac{2\pi\sqrt{|\sigma|}n}{K \log n} \text{ distinct points} \right\}.$$

Just like in (52), we have for arbitrary initial distribution

$$P(D_i) < C_1 \frac{\log \log n}{\log n}.$$

And so the probability that two such event occurs is smaller than

$$\left(C_4 \frac{K \log \log n}{\log n}\right)^2. \tag{60}$$

If at most one of the events D_i occurs and there is a pair (i', j') $1 \le i' \le j' \le K$ such that no C_{ij} occurs unless either i or j coincides with i' or j', then we have

$$L_2(n) > (K-3)\left(1 - \frac{\varepsilon}{2}\right) \frac{2\pi\sqrt{|\sigma|}n}{K \log n} - \frac{(K-3)(K-2)}{2} \frac{n \log \log n}{K \log^{1+\vartheta} n}.$$

Thus for large enough n, we have

$$L_2(n) > (1-\varepsilon) \frac{2\pi\sqrt{|\sigma|}n}{\log n}.$$

So by (59) and (60) we have

$$P\left(L_2(n) < (1-\varepsilon)\frac{2\pi\sqrt{|\sigma|}n}{\log n}\right) = O\left(\frac{\log^4 \log n}{\log^{2-2\vartheta} n}\right) = O\left(\frac{1}{\log^{2-3\vartheta} n}\right).$$

This together with (54) finishes the proof of the Lemma.

So the proof will be ready as soon as (55) and (58) are established. The idea of [11] is that in order to prove (55) and (58) it is useful to cut down the points of M_i which are visited in the extreme $\lceil n/\log^2 n \rceil$ steps. The number of these points can be roughly estimated, while the others are visited in steps quite far from each other and this will be enough for us. However, the precise arguments need some awkward computations.

Proof of (55) We introduce the notations

$$\alpha_{a,b} = \{\forall t = a, ..., b-1 : \eta_t \neq \eta_b\} \text{ and } \beta_{a,b} = \{\forall t = a+1, ..., b : \eta_a \neq \eta_t\}$$

which will be useful in the sequel. Following [11], we define

$$n_{(i,-)} = \lfloor (i-1)n/K \rfloor + \lceil n/\log^2 n \rceil \text{ and } n_{(i,+)} = \lfloor in/K \rfloor - \lceil n/\log^2 n \rceil .$$

A point, which is common in the paths $\eta_{n_{(i,-)}}, ..., \eta_{n_{(i,+)}}$ and $\eta_{n_{(j,-)}}, ..., \eta_{n_{(j,+)}}$ and not visited in the extreme $\lceil n/\log^2 n \rceil$ steps of M_i and M_j, has a pair of indices (k,l), $k \in n_{(i,-)}, ..., n_{(i,+)}$, $l \in n_{(j,-)}, ..., n_{(j,+)}$, such that it is visited at steps k and l, and it is not visited during steps $\lfloor (i-1)n/K \rfloor + 1, ..., k-1$, and steps $l+1, ..., \lfloor jn/K \rfloor$. So we have

$$\begin{aligned} E(M_{ij}) &\leq \frac{3n}{\log^2 n} + \sum_{k=n_{(i,-)}}^{n_{(i,+)}} \sum_{l=n_{(j,-)}}^{n_{(j,+)}} P\left(\alpha_{\lfloor (i-1)n/K \rfloor+1,k} \cap \{\eta_k = \eta_l\} \cap \beta_{l,\lfloor jn/K \rfloor}\right) \\ &\leq \frac{3n}{\log^2 n} + C_5 \sum_{k=n_{(i,-)}}^{n_{(i,+)}} \sum_{l=n_{(j,-)}}^{n_{(j,+)}} \frac{1}{\log(k - \lfloor (i-1)n/K \rfloor)} \frac{1}{l-k} \frac{1}{\log(\lfloor jn/K \rfloor - l)} \\ &\leq \frac{3n}{\log^2 n} + C_6 \frac{n}{K} \frac{\log n - \log\left(n/\log^2 n\right)}{\log^2 n} = O\left(\frac{n \log\log n}{K \log^2 n}\right). \end{aligned}$$

Note that we have used our estimations for the probability of avoiding the origin in some steps, visiting a new point, and returning to the origin, and these estimations are uniform in the initial distribution (with an appropriate C_5). Because the events whose intersection's probability is estimated above are dependent only via the internal states, it is obvious that the great order is uniform in i and j. So we arrived at (55).

Proof of (58) The dullest computation is the proof of formula (58). Let us prove

$$\sup_{1 \leq i < i' < j' < j \leq K} E(M_{ij} M_{i'j'}) = O\left(\frac{n^2 (\log\log n)^2}{K^2 \log^4 n}\right).$$

Let us introduce the notation $\mathcal{L}$ for the set of (k, k', l', l) such that

$$n_{(i,-)} \leq k \leq n_{(i,+)},\ n_{(i',-)} \leq k' \leq n_{(i',+)},\ n_{(j',-)} \leq l' \leq n_{(j',+)},\ n_{(j,-)} \leq l \leq n_{(j,+)}.$$

As it was mentioned before, we estimate the number of pair of points one of which is visited in either extreme $\lceil n/\log^2 n \rceil$ steps of M_i or M_j in a very obvious manner. The other pairs of lattice points (x and y, say) have a (k, k', l', l) element of $\mathcal{L}$, such that x is visited at step k but not visited during $\lfloor (i-1)n/K \rfloor + 1, ..., k-1$, and it is visited again at step k' but not visited during $k'+1, ..., \lfloor i'n/K \rfloor$; while y is visited at step l' but not visited during $\lfloor (j'-1)n/K \rfloor + 1, ..., j'-1$, and it is visited again at step l but not visited during $l+1, ..., \lfloor jn/K \rfloor$. So we have

$$E\left(M_{ij}M_{i'j'}\right) = O\left(\frac{n^2 \log\log n}{K \log^4 n}\right) + \sum_{M \in [-n,n]^2} \sum_{(k,k',l',l) \in \mathcal{L}} P\left(\mathcal{A}\right), \tag{61}$$

where

$$\begin{aligned} \mathcal{A} \;&=\; \alpha_{\lfloor (i-1)n/K \rfloor + 1, k} \cap \{\eta_{k'} - \eta_k = M\} \cap \beta_{k', \lfloor i'n/K \rfloor} \cap \{\eta_{l'} - \eta_{k'} = (0,0)\} \\ &\cap \alpha_{\lfloor (j'-1)n/K \rfloor + 1, l'} \cap \{\eta_l - \eta_{l'} = -M\} \cap \beta_{l, \lfloor jn/K \rfloor}. \end{aligned}$$

Denote the seven events, whose intersection is $\mathcal{A}$, by $\mathcal{A}_1, ..., \mathcal{A}_7$. Observe that for every $2 \le m \le 7$ the probability of $\mathcal{A}_m$ under the condition $\mathcal{A}_1 \cap ... \cap \mathcal{A}_{m-1}$ is just the probability of $\mathcal{A}_m$ with an appropriate initial distribution of ε. As we have uniform estimations in the initial distribution, we will be able to use them. Proposition 2 yields the existence of $a > 0$ (which depends only on the RWwIS), such that

$$\begin{aligned} P\left(\eta_k - \eta_0 = M\right) \;&<\; C_7 \exp\left(-\frac{a}{2k} M^T M\right) \left[\frac{1}{k} + \frac{1}{k^{3/2}}\right] + \frac{C_7}{k^2} \\ &<\; C_8 \left(\frac{1}{k} \exp\left(-\frac{a}{2k} M^T M\right) + \frac{1}{k^2}\right). \end{aligned}$$

So we have arrived at

$$P(\mathcal{A}_1 \cap \mathcal{A}_2) < C_9 \frac{1}{\log n} \left(\frac{1}{k'-k} \exp\left(-\frac{a}{2(k'-k)} M^T M\right) + \frac{1}{(k'-k)^2}\right), \tag{62}$$

and

$$P(\mathcal{A}_6 \cap \mathcal{A}_7 | \mathcal{A}_1 \cap ... \cap \mathcal{A}_5) < C_9 \frac{1}{\log n} \left(\frac{1}{l-l'} \exp\left(-\frac{a}{2(l-l')} M^T M\right) + \frac{1}{(l-l')^2}\right). \tag{63}$$

Consider the following factorization

$$P(\mathcal{A}) = P(\mathcal{A}_1 \cap \mathcal{A}_2) P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2) P(\mathcal{A}_6 \cap \mathcal{A}_7 | \mathcal{A}_1 \cap ... \cap \mathcal{A}_5), \tag{64}$$

and observe that

$$\sum_{k',l'} P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2) < C_{10} E(M_{i'j'}) = O\left(\frac{n \log\log n}{K \log^2 n}\right).$$

So we have to take the product of the expressions in (62), (63) and $P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2)$ and sum them up in all of the four indices to estimate (61). First, let us consider the product of the first terms in (62) and (63). We have to estimate

$$\sum_{M\in[-n,n]^2} \sum_{(k,k',l',l)\in\mathcal{L}} \exp\left(-\frac{a}{2}\left(\frac{1}{k'-k}+\frac{1}{l-l'}\right)M^TM\right)\frac{P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2)}{(k'-k)(l-l')\log^2 n}.$$

Using the fact

$$\sup_{d>a}\frac{1}{d}\sum_{M\in\mathbb{Z}^2}\exp\left(-\frac{a}{2d}M^TM\right) < +\infty \tag{65}$$

it suffices to estimate

$$\begin{aligned} &\sum_{(k,k',l',l)\in\mathcal{L}} \frac{(k'-k)(l-l')}{(l-l')+(k'-k)}\frac{P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2)}{(k'-k)(l-l')\log^2 n} \\ \leq\ & \frac{1}{\log^2 n}\sum_{(k,k',l',l)\in\mathcal{L}} \frac{P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2)}{(l-n_{(j',+)})+(n_{(i',-)}-k)} \end{aligned}$$

Using the already mentioned observation, i.e. that the condition $\mathcal{A}_1 \cap \mathcal{A}_2$ has contribution to the $P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5)$ only via $\varepsilon_{k'}$ and we have uniform estimates in the initial distribution, we conclude

$$\sum_{k',l'} P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2) < C_8 E\left(M_{ij}\right).$$

So it remains to estimate

$$\frac{1}{\log^2 n}E\left(M_{ij}\right)\sum_{k,l}\frac{1}{(l-n_{(j,-)})+(n_{(i,+)}-k)+2n/\log^2 n-1}$$

and it is just

$$O\left(\frac{n^2\left(\log\log n\right)^2}{K^2\log^4 n}\right)$$

uniformly in i and j, by an elementary computation.

Now, let us consider the product of the first term in (62) and the second term in (63) (the product of the second term in (62) and the first term in (63) can be estimated equivalently). In this case the easier estimation

$$P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2) < C_{11}\frac{1}{l'-k'} \tag{66}$$

will be enough. Thus our aim is to estimate

$$\frac{1}{\log^2 n}\sum_{(k,k',l',l)\in\mathcal{L}}\sum_{M\in[-n,n]^2}\exp\left(-\frac{a}{2\left(k'-k\right)}M^TM\right)\frac{1}{(k'-k)(l-l')^2(l'-k')}.$$

As above, we use (65) to handle the exponential terms. So the following estimation is enough for our purposes

$$\begin{aligned}\frac{1}{\log^2 n}\sum_{(k,k',l',l)\in\mathcal{L}}\frac{1}{(l-l')^2(l'-k')} &\leq \frac{1}{\log^2 n}\frac{n^4}{K^4}\frac{\log^4 n}{n^2}\frac{\log^2 n}{n} \\ &= O\left(\frac{n^2\left(\log\log n\right)^2}{K^2\log^4 n}\right).\end{aligned}$$

Our last task is to estimate the product of the second term in (62) and (63). The previous estimation (66) and

$$\begin{aligned}\frac{1}{\log^2 n}\sum_{(k,k',l',l)\in\mathcal{L}}\sum_{M\in[-n,n]^2}\frac{1}{(k'-k)^2(l-l')^2(l'-k')} &\leq n^6\frac{1}{\log^2 n}\frac{\log^4 n}{n^2}\frac{\log^4 n}{n^2}\frac{\log^2 n}{n} \\ &= O\left(\frac{n^2\left(\log\log n\right)^2}{K^2\log^4 n}\right)\end{aligned}$$

yield the required estimation.

A modified version of the proof presented above can be repeated for indices $1\leq i< i'<j<j'\leq K$. So we have finished the proof of formula (58). ■

6 Visited points in one dimension

Investigating the one dimensional case is not as important as the higher dimensions, as Lorentz processes used to be examined mainly in higher dimensions. However, one dimension is also interesting, as we will see some new features. We need some different means from the previous ones to prove asymptotics for $E_2\left(n\right)$, namely Tauberian arguments. Let us see the details.

Proposition 6 *For a one dimensional RWwIS fulfilling our basic assumptions with $\varepsilon_0\sim\mu$ we have*

$$\gamma_1(n)\sim\sqrt{\frac{2\left|\sigma\right|}{\pi}}*n^{-1/2}$$

Proof. Just like in the higher dimensional cases we consider the renewal equation for the reversed walk

$$\sum_{k=0}^{n} U_k \cdot R_{n-k} = \underline{1}.$$

Now, from row i we obtain

$$\sum_{j=1}^{s} \sum_{k=0}^{n} u_{ij}(k)\, x^k R_j(n-k)\, x^{n-k} = x^n. \tag{67}$$

Let us introduce the notations

$$\begin{aligned} \sum_{k=0}^{\infty} u_{ij}(k)\, x^k &= \alpha_{ij}(x) \\ \sum_{k=0}^{\infty} R_j(k)\, x^k &= \beta_j(x) \\ \sum_{k=0}^{\infty} x^k &= \omega(x). \end{aligned}$$

Obviously, these power series are convergent for $0 \leq x < 1$. In these terms, (67) means

$$\sum_{j=1}^{s} \alpha_{ij} \beta_j = \omega. \tag{68}$$

In order to obtain the order of the coefficients of $\gamma_1(n) = \sum_{j=1}^{s} \mu_j R_j(n)$ we use a Tauberian theorem which may be found in [5] (Theorem XIII.5). According to this we have

$$\omega(x) \sim \frac{1}{1-x}, \qquad x \to 1-. \tag{69}$$

For the coefficients of α_{ij}

$$\sum_{k=0}^{n} u_{ij}(k) \sim 2 \frac{1}{\sqrt{2\pi |\sigma|}} \mu_j n^{1/2}.$$

So, using the Tauberian theorem we infer

$$\alpha_{ij}(x) \sim 2 \frac{1}{\sqrt{2\pi |\sigma|}} \mu_j \Gamma\left(\frac{3}{2}\right) * \frac{1}{(1-x)^{1/2}}, \qquad x \to 1-. \tag{70}$$

From (68) we obtain

$$\sum_{j=1}^{s} \frac{\alpha_{ij}}{\alpha_{ii}} \beta_j = \frac{\omega}{\alpha_{ii}}. \tag{71}$$

Now, (70) yields

$$\frac{\alpha_{ij}(x)}{\alpha_{ii}(x)} \to \frac{\mu_j}{\mu_i}, \qquad x \to 1-. \tag{72}$$

Whence

$$\sum_{j=1}^{s}\mu_j\beta_j(x) \sim \frac{\sqrt{2\pi|\sigma|}}{2\Gamma\left(\frac{3}{2}\right)} * \frac{1}{(1-x)^{1/2}}, \qquad x \to 1- .$$

Since $\sum_{j=1}^{s}\mu_j R_j(k)$ is monotonic in k, using the mentioned Tauberian theorem we conclude

$$\gamma_1(n) = \sum_{j=1}^{s}\mu_j R_j(k) \sim \frac{\sqrt{2\pi|\sigma|}}{2\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} n^{-1/2} = \sqrt{\frac{2|\sigma|}{\pi}} n^{-1/2}$$

■

Corollary 3 *With arbitrary distribution of ε_0 the following holds*

$$E_1(n) \sim \sqrt{\frac{8|\sigma|}{\pi}} n^{1/2}.$$

Proof. From Proposition 6 the assertion immediately follows in the case of $\varepsilon_0 \sim \mu$. The proof is neither difficult in the case of arbitrary initial distribution. Analogously to (12) we have

$$\widetilde{\gamma}^{e_j}(n) = \sum_{k=1}^{s}\mu_k\widetilde{\gamma}^{e_k}(n-K) + \sum_{k=1}^{s} b_k^j(K)\widetilde{\gamma}^{e_k}(n-K) + O\left(K \cdot (n-K)^{-\frac{1}{2}}\right).$$

Taking $K = \lfloor\sqrt{n}\rfloor$ and dividing by $\sqrt{n}$ we find that the right hand side tends to $\sqrt{\frac{8|\sigma|}{\pi}}$ as $n \to \infty$. Hence the statement. ■

So, we have ascertained the asymptotic behavior of $E_d(n)$ in each dimension. An interesting feature of the one dimensional case is that weak law of large numbers holds, while strong law of large numbers fails to hold even in the case of SSRW. For the latter statement the reader is referred to [12] Chapter 20 Remark 1. For the previous statement observe that $V_2(n) = o(n)$ holds, which can be easily verified by repeating a simplified version of the proof of Theorem 4, and Chebyshev's inequality applies.

7 First return, locally perturbed walks

Now, we are interested in a question based on a problem treated by Spohn in [14]. The problem is the following. Consider a locally perturbed SSRW which behaves like an ordinary SSRW except in the origin. When the particle arrives at the origin it continues in the direction it came from. The interesting question is the decay of the autocorrelation function, namely

$$E\left(\langle X_1, X_n\rangle\right),$$

where X_n is the n^{th} step. Now, we would like to examine the same question for RWwIS. One main point here is to determine the probability of the event that a RWwIS returns to the origin at time n for the first time. Let us denote this probability by $f_d^{\nu}(n)$ (where $\varepsilon_0 \sim \nu$) with the remark that recall this quantity depends on the parameters of our RWwIS, but we are mainly interested in the asymptotics depending on the dimension (just like in the case of $E_d(n)$). For a simple random walk (i.e. each steps are independent) with zero mean and finite variance V it is shown in [1] that

$$f_1(n) \sim \sqrt{\frac{V}{2\pi}} * n^{-3/2}.$$

First we prove sharp upper estimates for the decay of $f_d(n)$ for RWwIS. Surprisingly, in the literature I was unable to find a theorem concerning the decay of $f_d(n)$ in higher dimensions, even in the case of SSRW.

Theorem 9 *For a RWwIS fulfilling our basic assumptions [and assuming that (1) exists in the two dimensional case] with arbitrary ν distribution of ε_0*

$$f_1^{\nu}(n) = O\left(n^{-3/2}\right), \quad f_1^{\nu}(n) \neq o\left(n^{-3/2}\right) \tag{73}$$

$$f_2^{\nu}(n) = O\left(\frac{1}{n\log^2(n)}\right), \quad f_2^{\nu}(n) \neq o\left(\frac{1}{n\log^2(n)}\right) \tag{74}$$

$$f_d^{\nu}(n) = O\left(n^{-d/2}\right), \quad f_d^{\nu}(n) \neq o\left(n^{-d/2}\right) \quad \textit{for } d \geq 3 \tag{75}$$

Proof. First of all, observe that proving the statement for $\nu = \mu$ would be enough, as since our basic assumption (i) all component of μ are positive.

The simplest assertion is (75), so we handle it first. We have seen in Section 3 that for $\gamma_d(n)$ of the reversed random walk

$$\sum_{k=1}^{n} f_d(k) = 1 - \gamma_d(n) = 1 - \gamma_d + O\left(n^{1-d/2}\right)$$

From the proof of Theorem 3 it follows that the great order on the right hand side is sharp. This yields $f_d(n) \neq o\left(n^{-d/2}\right)$. The other assertion is trivial as $f_d(n) \leq P\left(\eta_n = 0\right) = O\left(n^{-d/2}\right)$.

For the lower dimensions we generalize an argument in [1]. Define

$$Q_d^n(x,i,y,j) = P\left(\xi_n = (y,j), \eta_k \neq 0, \forall 1 \leq k < n | \xi_0 = (x,i)\right),$$

where d is the dimension.

Let $n = 3m$ and $1 \leq i \leq m$. The cases $n = 3m \pm 1$ can be treated the same way.

$$\begin{aligned} f_d^{e_i}(n) &= \sum_{l=1}^{s} Q_d^n(0,i,0,l) = \sum_{y,z\neq 0} \sum_{j,k,l=1}^{s} Q_d^m(0,i,y,j) Q_d^m(y,j,z,k) Q_d^m(z,k,0,l) \\ &\leq \sup_{y,z,j,k} Q_d^m(y,j,z,k) P(\eta_k \neq 0, \forall 1 \leq k < m | \xi_0 = (0,i)) \sum_{z\neq 0} \sum_{k,l=1}^{s} Q_d^m(z,k,0,l) \end{aligned}$$

From the local limit theorem it follows that

$$\sup_{y,z,j,k} Q_1^m(y,j,z,k) = O(m^{-1/2})$$

and

$$\sup_{y,z,j,k} Q_2^m(y,j,z,k) = O(m^{-1}).$$

Proposition 6 yields $P(\eta_k \neq 0, \forall 1 \leq k < m | \xi_0 = (0,i)) = O(m^{-1/2})$ in the one dimensional case, while (35) yields $O\left(\frac{1}{\log m}\right)$ for the same quantity in the two dimensional case. So, for the upper estimates it suffices to prove

$$\sum_{z\neq 0} \sum_{k,l=1}^{s} Q_1^m(z,k,0,l) = O(m^{-1/2}), \tag{76}$$

$$\sum_{z\neq 0} \sum_{k,l=1}^{s} Q_2^m(z,k,0,l) = O\left(\frac{1}{\log m}\right). \tag{77}$$

In order to prove (76) and (77) we use the reversed walk, again. (9) yields that for all $((0,i_1),(y_1,i_2),(y_1+y_2,i_3),...,(y_1+y_2+...+y_{m-1},i_m))$ trajectories

$$\mu_{i_1} p_{y_1,i_1,i_2} \mu_{i_2} p_{y_2,i_2,i_3} \ldots \mu_{i_{m-1}} p_{y_{m-1},i_{m-1},i_m} = \mu_{i_2} q_{-y_1,i_2,i_1} \mu_{i_3} q_{-y_2,i_3,i_2} \ldots \mu_{i_m} q_{-y_{m-1},i_m,i_{m-1}},$$

where the factors $\mu_{i_2},...,\mu_{i_{m-1}}$ drop out. Thus

$$\sum_{z\neq 0} \sum_{k,l=1}^{s} Q_d^m(z,k,0,l) \leq \max_{1\leq i,j\leq s} \frac{\mu_i}{\mu_j} \sum_{z\neq 0} \sum_{k,l=1}^{s} \widetilde{Q}_d^m(0,l,z,k), \tag{78}$$

where $\widetilde{Q}$ is the same object as Q defined for the reversed walk. The right hand side of (78) can be bounded by some constant times the probability of the event that the stationary reversed walk does not return to the origin in the first m steps, which is $O(m^{-1/2})$ in one dimension and $O\left(\frac{1}{\log m}\right)$ in two dimensions. Thus we arrived at (76) and (77).

Now, suppose that $f_2^\mu(n) = o\left(\frac{1}{n\log^2(n)}\right)$ or $f_1^\mu(n) = o\left(n^{-3/2}\right)$. Thus

$$1 - \gamma_1(n) = \sum_{k=1}^{n} f_1(k) = c_1 + o(n^{-1/2}),$$

contradicting to Proposition 6, or

$$1 - \gamma_2(n) = \sum_{k=1}^{n} f_2(k) = c_2 + o\left(\frac{1}{\log n}\right)$$

contradicting to (35). ■

The application of the above theorem for the question of the decay of the autocorrelation function is only clear in a very special case, i.e. in the one dimensional closest neighbor model. Assuming that $d = 1$, and that $p_{y,i,j} = 0$ unless $|y| = 1$, obviously, the only contribution of the perturbation comes from paths where the particle is at the origin at time n. So we have

$$E\left(\langle X_1, X_n\rangle\right) = -f_1^{\nu_1}(n) + \left(f_1^{\nu_1} * f_1^{\nu_2}\right)(n) - ...$$

with some $\nu_1, \nu_2, ...$ distributions. Now, Theorem 9 implies $E\left(\langle X_1, X_n\rangle\right) = O\left(n^{-3/2}\right)$. An interesting topic would be the investigation of this autocorrelation function in higher dimensions.

8 Distribution at the time of hitting the origin

This section concerns a problem which has no analogue in the theory of SSRW. We are interested in the distribution of ε when the RWwIS hits the origin for the first time. Two cases will be treated. In the first case we will assume that the RWwIS starts from very far away. In [15] it is proved that in two dimensions $\exists \rho \in [0,1]^d$ probability distribution, such that for all $1 \le i \le s$

$$\rho_j = \lim_{|x|\to\infty} P\left(\exists n : \xi_n = (0,j), 1 \le k < n : \eta_k \neq 0 | \xi_0 = (x,i)\right).$$

Roughly speaking, it means that the distribution at the time of hitting the origin does not depend on the starting place and state as long as it is quite far away. We will see that it is not true in one dimension, it will be important whether we start from minus, or plus infinity. In the second case the starting point will be the origin itself. Similar questions for Lorentz process are treated in [2].

8.1 Analytical investigation

Consider a simple one dimensional RWwIS, i.e. $p_{y,i,j} = 0$ unless $y = \pm 1$. Denote $Q_1 = (p_{1,i,j})_{i,j=1,\dots,s}$ $Q_{-1} = (p_{-1,i,j})_{i,j=1,\dots,s}$. Obviously, we have $Q = Q_1 + Q_{-1}$. In this case basic assumption (iii) means

$$\left\langle \mu^T Q_1, 1 \right\rangle = \left\langle \mu^T Q_{-1}, 1 \right\rangle = \frac{1}{2}$$

From the local limit theorem it follows that the walk starting from $n \in \mathbb{Z}$ reaches the origin almost surely. Define the matrix $P = (p_{i,j})_{i,j=1}^{s}$ the following way

$$P(\varepsilon_\tau = j | \varepsilon_0 = i) = p_{ij},$$

where

$$\tau = \min \{k \geq 0 | \eta_k = -1\}.$$

From the definition of P it immediately follows that

$$P = Q_{-1} + Q_1 P^2.$$

It is clear that the for ρ stationary distribution of P

$$\rho_i = \lim_{n->\infty} P(\exists k : \xi_k = (0,i), \eta_0 = n, \forall 1 \leq l < k : \eta_l \neq 0).$$

The above argument can be repeated in the case of starting point at $(-n)$, when $n \in \mathbb{N}$ is large. Let us define $P = (p_{i,j})_{i,j=1}^{s}$ the following way

$$P(\varepsilon_{\widetilde{\tau}} = j | \varepsilon_0 = i) = p_{ij},$$

where

$$\widetilde{\tau} = \min \{k \geq 0 | \eta_k = 1\}.$$

From the definition of $\widetilde{P}$ it immediately follows that

$$\widetilde{P} = Q_1 + Q_{-1}\widetilde{P}^2.$$

and with the notation $\widetilde{\rho}$ for the stationary distribution of $\widetilde{P}$

$$\widetilde{\rho}_i = \lim_{n->\infty} P\left(\exists k : \xi_k = (0,i), \eta_0 = -n, \forall 1 \leq l < k : \eta_l \neq 0\right).$$

The most fascinating phenomenon is that μ, ρ and $\widetilde{\rho}$ can be all different.

Now, let us treat the case when $\eta_0 = 0$, and we are interested in the distribution of ε_T, where

$$T = \min\left\{k > 0 : \eta_k = 0\right\}.$$

Obviously, there is a matrix $R = (R_{i,j})_{i,j=1}^s$ such that

$$P\left(\varepsilon_T = j | \varepsilon_0 = i\right) = R_{ij}.$$

And the simplicity of the RWwIS yields

$$R = Q_1 P + Q_{-1}\widetilde{P}.$$

A not surprising result is that μ is stationary for R.

Proposition 7 *For a one or two dimensional RWwIS fulfilling our basic assumptions we have*

$$\mu^T R = \mu^T,$$

and R is irreducible.

Proof. Let $1 \leq i, j \leq s$ be arbitrary internal states. With the notation$i_0 = i$, $i_m = j$ consider a trajectory $((0, i_0), (y_1, i_1), (y_1 + y_2, i_2), ..., (y_1 + y_2 + ... + y_m, i_m))$, for which

$$\begin{aligned} y_1 + y_2 + ... + y_m &= 0, \\ y_1 + y_2 + ... + y_k &\neq 0 \quad 1 \leq k \leq m-1 \end{aligned}$$

and $1 \leq i_1, ..., i_{m-1} \leq s$ are internal states. As in the proof of Theorem 9, (9) yields

$$\mu_{i_0} p_{y_1,i_0,i_1} \mu_{i_1} p_{y_2,i_1,i_2} ... \mu_{i_{m-1}} p_{y_m,i_{m-1},i_m} = \mu_{i_1} q_{-y_1,i_1,i_0} \mu_{i_2} q_{-y_2,i_2,i_1} ... \mu_{i_m} q_{-y_m,i_m,i_{m-1}}. \quad (79)$$

As (79) holds for all trajectories avoiding the origin except for the endpoint, we conclude

$$\mu_i R_{ij} = \mu_j \widetilde{R}_{ji}, \tag{80}$$

where $\widetilde{R} = \left(\widetilde{R}_{i,j}\right)_{i,j=1}^{s}$ is the matrix defined for the reversed walk as the same way R was defined for the general one. (80) yields

$$\sum_{i=1}^{s} \mu_i R_{ij} = \sum_{i=1}^{s} \mu_j \widetilde{R}_{ji} = \mu_j,$$

where we used that $\widetilde{R}$ is a stochastic matrix.

To see the irreducibility, observe that the local limit theorem yields

$$P(\xi_n = (0,j) | \xi_0 = (0,i)) > 0,$$

for $\forall 1 \leq i, j \leq s$ with an appropriate n. So there is a k such that $\left(R^k\right)_{i,j} > 0$. Thus we have arrived at the assertion. ■

8.2 Some examples

8.2.1 example

Let $Q = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $Q_1 = \begin{pmatrix} \frac{2}{5} & \frac{5}{12} \\ \frac{3}{10} & \frac{13}{80} \end{pmatrix}$, $Q_{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{4} \\ \frac{1}{5} & \frac{27}{80} \end{pmatrix}$, $\mu = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}$. Using Mathematica we obtain

$$P = \begin{pmatrix} \frac{-1597+19\sqrt{14961}}{1736} & \frac{3333-19\sqrt{14961}}{1736} \\ \frac{2}{217}\left(164-\sqrt{14961}\right) & \frac{1}{217}\left(-111+2\sqrt{14961}\right) \end{pmatrix},$$

$$\widetilde{P} = \begin{pmatrix} \frac{1}{14}\left(127-\sqrt{14961}\right) & \frac{1}{14}\left(-113+\sqrt{14961}\right) \\ \frac{1}{21}\left(-111+\sqrt{14961}\right) & \frac{1}{21}\left(132-\sqrt{14961}\right) \end{pmatrix},$$

$$R = \begin{pmatrix} \frac{1}{210}\left(210-\sqrt{14961}\right) & \frac{\sqrt{14961}}{210} \\ \frac{\sqrt{14961}}{280} & \frac{1}{280}\left(280-\sqrt{14961}\right) \end{pmatrix},$$

$\rho = \begin{pmatrix} a & 1-a \end{pmatrix}$, where

$$a = \frac{16\left(-164+\sqrt{14961}\right)}{7\left(-851+5\sqrt{14961}\right)}.$$

and $\widetilde{\rho} = \begin{pmatrix} b & 1-b \end{pmatrix}$, where

$$b = \frac{2\left(-111+\sqrt{14961}\right)}{-561+5\sqrt{14961}}$$

In this case $a - \mu_1 \approx -0,03$ while $b - \mu_1 \approx 0,02$. The unique stationary distribution of R is μ.

8.2.2 example (reversible RWwIS)

Let $Q = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, $Q_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{3} \end{pmatrix}$, $Q_{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{12} & \frac{1}{3} \end{pmatrix}$, $\mu = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

With the notation

$$c = -\frac{45-7\sqrt{66}}{6\left(-11+2\sqrt{66}\right)} \approx 0,38$$

we have $\rho = \begin{pmatrix} c & 1-c \end{pmatrix}$, and $\widetilde{\rho} = \begin{pmatrix} 1-c & c \end{pmatrix}$.

8.2.3 two dimensional example

Consider the RWwIS B_3 defined in Subsection 4.1. Simulations with 1000 trajectories starting from state 1 show that with 0.155 probability it takes more than 10^7 steps to return to the origin. Unfortunately, in these cases we cannot simulate the distribution at the time of first return, because it would take too much time. Under the assumption that the RWwIS returns in less than 10^7 steps we found that the probability of returning in state 1 is 0.25 while returning in state 2 is 0.75. These simulations suggest that $R = Q$ can be true.

9 Some further path properties

As we have seen in Section 3 and in Section 4, some interesting asymptotic path properties can radically differ in two and in any higher dimensional settings. Besides the number of distinct sites visited by the random walk, one can investigate the number of returns to the origin, or the number of sites visited exactly once up to time n. Basically, these are questions treated in [4] for SSRW. Now, we would like to extend this theory to the case of RWwIS. Let us begin with the number of returns to the origin. We generalize the arguments of [4].

If $d \geq 3$, obviously, this will be a finite number almost surely. Let $\mathcal{R}$ denote this random variable. In the case of SSRW we have

$$P(\mathcal{R} = k) = (1 - \gamma_d)^k \gamma_d,$$

so $\mathcal{R}$ has a geometric distribution.

In the case of RWwIS let us define the matrix $(S_{i,j})_{i,j=1}^s$

$$S_{i,j} = P(\varepsilon_T = j | \varepsilon_0 = i),$$

which is just the probability of the event that the random walk returns to the origin in internal state j, assuming that it starts from internal state i (recall that $T = \min\{k > 0 : \eta_k = 0\}$). Because of the transiency we have

$$0 < w_i := 1 - \sum_{j=1}^{s} S_{i,j} < 1.$$

Now it is easy to see that for $k \geq 0$

$$P(\mathcal{R} = k | \varepsilon_0 \sim \nu) = \nu^T S^k w,$$

where $w = (w_1 ... w_s)$. It is not a geometric distribution, but also has an exponential decay, where the exponent is the largest eigenvalue of S (which is smaller than one because of the transiency).

The two dimensional case is much more involved. For simplicity, assume $\varepsilon_0 \sim \mu$. Let $\mathcal{R}_n$ be the number of returns to the origin up to time n. Define

$$\mathcal{T}_n = \frac{\mathcal{R}_n}{\log n}.$$

We can write the excursion lengths the following way. Let $\delta_1, \delta_2, ...$ be a Markov chain, where δ_k is the pair of two internal states: the internal state at the time of k^{th} and $(k+1)^{th}$ visit to the origin (where the time 0 is called the first visit). Obviously,

$$P(\delta_k = (i_1, i_2) | \delta_{k-1} = (i_0, i_1)) = R_{i_1, i_2},$$

and we use Proposition 7 to infer that the stationary distribution of δ is

$$P(\delta_k = (i_1, i_2)) = \mu_{i_1} R_{i_1, i_2}$$

Now let $Z_1, Z_2, \ldots$ denote the lengths of the excursions, i.e.

$$\begin{aligned} Z_1 &= \min\{k > 0 : \eta_k = 0\}, \\ Z_l &= \min\{k > Z_{l-1} : \eta_k = 0\} - Z_{l-1}. \end{aligned}$$

Observe that $Z_1, Z_2, \ldots$ are not independent, but they are conditionally independent if we know $\delta_1, \delta_2, \ldots$. Let $x > 0$, and let us try to estimate $P(\mathcal{T}_n < x)$. Define the integer q the following way

$$q = \lfloor x \log n \rfloor + 1.$$

Obviously, if $Z_1 + \ldots + Z_q < n$ then $\mathcal{T}_n \geq x$. Now, if $Z_1, Z_2, \ldots$ were independent, we would be able to estimate $P(\mathcal{T}_n < x)$ easily, just like in [4]. Observe that because of the irreducibility of R and the ergodic theorem of Markov chains we will have $\#\{1 \leq l \leq k : \delta_l = (i_1, i_2)\} \approx \mu_{i_1} R_{i_1, i_2} k$. Furthermore, as Z_k is independent of the other Z_l's given δ_k it is plausible that the distribution of $Z_1 + \ldots + Z_q$ will be close to the one of the sum of q independent copies of Z_1. The latter is easily estimated just like, as it was mentioned, in [4]. Let us formulate the above idea.

Let $(W_1, \beta_1), (W_2, \beta_2), \ldots$ be i.i.d. random variables, with the same distribution as (Z_1, δ_1). Because of the weak law of large numbers, with fix $\varepsilon > 0$ for large enough n and for all $1 \leq i, j \leq s$ the following holds

$$P\left(\#\{1 \leq l \leq q : \beta_l = (i,j)\} > (1-\varepsilon)\mu_i R_{i,j} q\right) > 1 - \varepsilon.$$

Thus

$$\left[P\left(W_1 < \frac{n}{q}\right)\right]^q \leq \varepsilon + \prod_{i,j=1}^{s} \left[P\left(W_1 < \frac{n}{q} | \beta_1 = (i,j)\right)\right]^{(1-\varepsilon)\mu_i R_{i,j} q}. \tag{81}$$

On the other hand,

$$\begin{aligned} P(\mathcal{T}_n \geq x) &\geq P(Z_1 + \ldots + Z_q < n) = E\left(E\left(1_{[Z_1 + \ldots + Z_q < n]} | \delta_1, \ldots \delta_q\right)\right) \\ &\geq E\left(\prod_{l=1}^{q} E\left(1_{[Z_l < \frac{n}{q}]} | \delta_l\right)\right) \\ &= E\left(\prod_{i,j=1}^{s} \left[P\left(Z_1 < \frac{n}{q} | \delta_1 = (i,j)\right)\right]^{\#\{1 \leq l \leq q : \delta_l = (i,j)\}}\right). \end{aligned}$$

As it was mentioned, we use the irreducibility of R and the ergodic theorem of Markov chains to infer

$$P(\#\{1 \leq l \leq q : \delta_l = (i,j)\} < (1+\varepsilon)\mu_i R_{i,j} q) > 1 - \varepsilon.$$

Thus

$$P\left(\mathcal{T}_n \geq x\right) \geq (1-\varepsilon) \prod_{i,j=1}^{s} \left[P\left(Z_1 < \frac{n}{q} | \delta_1 = (i,j)\right)\right]^{(1+\varepsilon)\mu_i R_{i,j} q}. \tag{82}$$

Now (81) and (82) yields

$$\left(P\left(\mathcal{T}_n \geq x\right) \frac{1}{1-\varepsilon}\right)^{1-2\varepsilon} + \varepsilon \geq \left[P\left(W_1 < \frac{n}{q}\right)\right]^q.$$

Now

$$P\left(W_1 < \frac{n}{q}\right) = 1 - \gamma_2\left(\frac{n}{q}\right)$$

implies

$$\left[P\left(W_1 < \frac{n}{q}\right)\right]^q = \exp(-2\pi\sqrt{|\sigma|}x) + o(1).$$

So we have arrived at

$$P\left(\mathcal{T}_n \geq x\right) \geq \exp(-2\pi\sqrt{|\sigma|}x) + o(1).$$

For the other direction observe that if there is an m index, $m < q-2$ such that $Z_m > n$, then $\mathcal{T}_n < x$. Thus

$$P\left(\mathcal{T}_n < x\right) > 1 - P(Z_1 < n, ..., Z_{q-2} < n).$$

Since

$$\left[P\left(W_1 < n\right)\right]^{q-2} = \exp(-2\pi\sqrt{|\sigma|}x) + o(1),$$

a very similar argument to the previous one leads to

$$P\left(\mathcal{T}_n < x\right) \geq 1 - \exp(-2\pi\sqrt{|\sigma|}x) + o(1).$$

Thus we have arrived at

Theorem 10 *For a two dimensional RWwIS fulfilling our basic assumptions and the convergence of (1) we have*

$$\lim_{n\to\infty} P(\mathcal{R}_n < x\log n) = 1 - \exp(-2\pi\sqrt{|\sigma|}x).$$

Now, let us treat the second problem which was mentioned at the beginning of this Section. Let $M(n)$ denote the number of lattice points on the plane which are visited by a RWwIS exactly once up to time n. In [4] it is proved that in the case of SSRW

$$E(M(n)) = \frac{n\pi^2}{\log^2 n} + O\left(\frac{n \log\log n}{\log^3 n}\right). \tag{83}$$

As the analogue of (83) under our basic assumptions has not been calculated yet, we show two interesting examples.

Example 11 *Let $s = 2$, $0 < a < 1$ and*

$$Q = \begin{pmatrix} a & 1-a \\ 1 & 0 \end{pmatrix}.$$

Moreover, let $p_{y,i,j} = 0$ unless $i = 2, j = 1$. In the latter case define the distribution of the step paying respect to our basic assumptions (i.e. zero mean, not restricted to some sub-lattice). Obviously, if the random walk visits a new place, it must be in internal state 1, and thus it remains in that very same place for the next time. So, $M(n)$ can be $0, 1$ or 2.

Example 12 *Let $s = 2, 0 < a, b < 1$ and*

$$Q = \begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}.$$

Moreover, let $p_{y,i,j} = 0$ unless $i = 2, j = 2$. In the latter case define the distribution of the step paying respect to our basic assumptions (i.e. zero mean, not restricted to some sub-lattice). Obviously, if the random walk is in state 1, it remains at its place for at least one step. Thus with the notation

$$R_n^{\nu} = P(0 < k \le n : \eta_k \ne \eta_0 | \varepsilon_0 \sim \nu)$$

$R_n^{(1,0)} = 0$. Moreover, we have seen that $R_n^{\mu} \sim \frac{2\pi\sqrt{|\sigma|}n}{\log n}$, so $R_n^{(0,1)} \sim \frac{2\pi\sqrt{|\sigma|}}{\mu_2}\frac{n}{\log n}$. Obviously, if the random walk visits a new site, it must be at internal state 2. So

$$E(M(n)) = \sum_{k=0}^{n} \gamma(k) R_{n-k}^{(0,1)}. \tag{84}$$

From (84),exactly as in [4],one can deduce

$$E(M(n)) = \frac{4\pi^2 |\sigma|}{\mu_2}\frac{n}{\log^2 n} + O\left(\frac{n \log\log n}{\log^3 n}\right).$$

While Example 11 shows that the behavior can be totally different as in the case of SSRW (in this "trivial" case, i.e. when we prohibit visiting a site only once), Example 12 shows that even in this "similar" case (i.e. the same order of decay) the involved constant depends on other quantities besides the variance.

10 Conclusion and final remarks

Conclusion

In my diploma thesis, I endeavored to reveal some interesting features of random walks in general. Generality means that we do not have to postulate the independence of the successive steps, but we can assume that they are connected by an underlying Markov chain. In a considerable part of this work, I have chosen to follow one particular thread. First, the already well known local limit theorem was sharpened with a remainder term and a further estimation. These generalizations allowed me to prove Dvoretzky-Erdős type theorems in the plane for RWwIS. At the same time, the one and the high dimensional cases were also dealt with, the latter of which is not discussed in the classical paper of Dvoretzky-Erdős. The asymptotic behavior of the probability of the first return to the origin was ascertained in every dimension. Note that for this statement, the asymptotic value of $E_d(n)$ was needed. Furthermore, the number of returns to the origin in $d \geq 2$ has been determined, which is a generalization of a result in [4].

Moreover, the auto-correlation function of some locally perturbed walks was considered. We succeeded in estimating this quantity in a very special case. For more general cases, some further ideas would be required. The distribution of the internal states by the time of hitting the origin was also discussed. In particular, it has been proved that the distribution is stationary at the time of the first return provided that it was stationary initially. We have also seen that in one dimension it is relevant whether we start from infinity or minus infinity.

Final remarks

1. Our asymptotic investigations show that RWwIS behaves like the simple symmetric random walk in an asymptotic sense. The main features are very similar, only the

involved constants differ. The results showing that the asymptotic behavior is independent from the initial distribution on the internal states (e.g. Proposition 4 and 5, Remark 3) are intuitively trivial as after some steps ε will be very close to μ. Nevertheless, these assertions need formal proofs as well, especially as they are used in the estimations of $V_d(n)$. Of course, this similarity to the simple symmetric random walk could change if the generalization were carried further, for instance, if a countable set of internal states was allowed. This model is not yet discussed, it must need some more involved technics.

2. Our basic assumption (ii) is not essential. The above theorems could be generalized to the case of dropping basic assumption (ii), as the limit theorem in [8] is proved for this case, as well. Only the computations would become longer. The other three assumptions are essential.

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