Lorentz process with infinite horizon and the martingale method

Péter Nándori, Domokos Szász, and Tamás Varjú

Abstract

We prove that for the planar periodic Lorentz process with infinite horizon, the super-diffusively rescaled trajectory cannot have any other weak limit, as the Brownian motion. Note that this was also proven in [12] and [5]. The novelty of our proof is the use of the martingale method à la Stroock and Varadhan [11], which was useful by the treatment of local perturbation in case of finite horizon [8]. We hope that this new proof can be useful by attacking local perturbation in case of infinite horizon.

1 Introduction

In 1905, Hendrik Lorentz [9] introduced Lorentz gas as a model of motion of electrons in a metal. By considering the dynamics of just one classical electron in a crystal one has the *periodic planar Lorentz process*. It is the $\mathbb{Z}^2$-extension of a *toric Sinai billiard* (i. e. of one with strictly convex smooth scatterers on the 2-torus). It is known that the limiting distribution of the rescaled displacement is Gaussian and that of the rescaled orbit is a Wiener process. The scaling, however, is either the diffusive $\sqrt{n}$ or the slightly super-diffusive $\sqrt{n \log n}$ depending on whether the billiard has finite or infinite *horizon*, resp. (we say that the horizon is finite if the free flight time is finite). In the first case the limiting covariance is given by the Green-Kubo formula (cf. [2], [3]). In the infinite horizon case, however, the stronger $\sqrt{n \log n}$ scaling suppresses time correlations and the limiting covariance has a simple form.
The central limit theorem in the super-diffusive case was conjectured by [1] and first established by [12] - with the method of Young towers. Recently, [5] gave a new proof - not just for the central limit theorem, but for the convergence to the Wiener process, too, which we call weak invariance principle (WIP). The method of [5] is the standard pair technique of Chernov and Dolgopyat combined with the so-called Bernstein's small block-big block technique from probability theory.
For non periodic Lorentz processes, though, much less is known. There is an archetype of such a result: [8] proves that if the periodicity of a Lorentz process with finite horizon is spoiled in a compact domain, then the WIP remains true. The proof of this result is

based on the standard pair technique, but the probabilistic ingredient is the martingale method of Stroock and Varadhan. Besides, the question of the WIP in locally perturbed periodic Lorentz process with infinite horizon remains open. There is some hope that the martingale method might be useful in attacking this problem, too. This is the motivation of our present work.
Here, we identify the possible limit points of the super-diffusively rescaled trajectory in periodic Lorentz processes (which is, of course, Wiener process, solely) with the use of the martingale method. This is almost the same, as giving a third proof for the WIP for periodic Lorentz process with infinite horizon - what missing, is checking that the weak limit indeed exists, i.e. proving tightness. We also stress the fact that our proof is strongly based on the one of [8] and also uses similar cut-offs (although not the same) as, and other technical results from [5].
In Section 2 we formulate our statement and provide the basic definitions and lemmas for its proof, while in Section 3, we present the actual proof.

2 Preliminaries

Here, we summarize very briefly the most important notions and notations from Sinai billiards needed in the present work. For a much ampler description, consult [7]. Define $\mathcal{D} = \mathbb{R}^2 \setminus \cup_{i=1}^{\infty} B_i$, where $B_1, \dots, B_k$ are disjoint strictly convex domains inside the unit torus, whose boundaries are C^3-smooth and whose curvatures are bounded from below. $B_{k+1}, B_{k+2}, ...$ are the translational copies of $B_1, \dots, B_k$ with translations in $\mathbb{Z}^2$. The billiard flow is the dynamics of a point particle in $\mathcal{D}$, which consists of free flight inside $\mathcal{D}$ and specular reflection on $\partial\mathcal{D}$. It is common to take the Poincaré section on the boundaries of the scatterer (billiard ball map). The phase space of the billiard ball map is

$$\mathcal{M} = \{x = (q, v) \in \partial\mathcal{D} \times S^1, \langle v, n \rangle \geq 0\},$$

where n is the normal vector of ∂D at the point q pointing inside $\mathcal{D}$, and the map itself is denoted by $\mathcal{F} : \mathcal{M} \to \mathcal{M}$. The natural invariant measure on $\mathcal{M}$, which we denote by μ, is the projection of the Lebesgue measure on the phase space of the billiard flow. In fact, $d\mu = const \cos\phi dr d\phi$, where r is the arc length parameter on $\partial\mathcal{D}$ and $\phi \in [-\pi/2, \pi/2]$ is the angle of v and n. We will write $q(x)$ for the the projection of the point x to its first coordinate (that is $q(x) \in \partial\mathcal{D}$). The free flight vector is $\Delta_0(x) = q(\mathcal{F}(x)) - q(x)$. We will also write $q_k(x) = q(\mathcal{F}^k(x))$ and $\Delta_k(x) = \Delta_0(\mathcal{F}^k(x)) = q_{k+1}(x) - q_k(x)$.
Analogously, one can define the Sinai billiard on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then one needs to introduce $\mathcal{D}_0 = \mathbb{T}^2 \setminus \cup_{i=1}^{k} B_i$, and define $\mathcal{M}_0$, $\mathcal{F}_0$ and μ_0 as before. Now μ is the periodic extension of μ_0. Since μ is infinite and μ_0 is finite, we choose the constant in the definition of μ such that μ_0 is a probability measure. The functions q_k and Δ_k can also be defined on $\mathcal{M}_0$ as the restriction of the corresponding functions on $\mathcal{M}$.
Hyperbolicity and ergodicity of $\mathcal{F}_0$ (nice properties) were proven by Sinai [10]. An unpleasant property of the billiard map is the presence of singularities (corresponding to

grazing collisions). An elegant and flexible approach to overcome this problem and prove statistical properties is the standard pair method developed by Chernov and Dolgopyat [6]. What follows, is an informal description of this method.
For almost every $x \in \mathcal{M}_0$, stable and unstable manifolds through x exist. There is a factor of stretching in the unstable direction, which is bounded from below by some $\Lambda > 1$. Nevertheless, these factors are not bounded from above (the closer is x to the grazing collisions, $\{\cos\phi = 0\}$, the stronger is the expansion), which makes difficult to control the distortion of unstable manifolds. That is why it is common to introduce the following additional (secondary) singularities

$$S_{\pm k} = \{(r, \phi) : \phi = \pm\pi/2 \mp k^{-2}\}$$

for k larger than some k_0, yielding bounded distortion of an unstable manifold disjoint to all singularities. An unstable curve is some curve $W \subset \mathcal{M}$ such that at every point $x \in W$, the tangent space $T_x W$ is in the unstable cone (slightly weaker property than the unstable manifold). Further, W is homogeneous, if does not intersect any singularity. A pair $l = (W, \rho)$ is called a standard pair, if W is a homogeneous unstable curve and ρ is a regular probability density supported on W. In order to define the desired regularity of the density ρ, we need one more notion. For two points x, y on W, we write $s_{\pm}(x, y)$ for the smallest integer n such that $\mathcal{F}_0^{\pm n}(x)$ and $\mathcal{F}_0^{\pm n}(y)$ are separated by some singularity curve. A function f is called dynamically Hölder continuous, if there exists sone $\theta_f < 1$ such that for any x and y lying in some unstable (stable, resp.) curve W, the following inequality holds

$$|f(x) - f(y)| < K_f \theta_f^{s_{\pm}(x,y)}.$$

Now, the regularity property required for the density ρ is that $\log\rho$ should be dynamically Hölder continuous. For a standard pair $l = (W, \rho)$, we write $\mathbb{E}_l$ for the integral with respect to ρ, $\mathbb{P}_l(A) = \rho(A)$ and $length(l) = length(W)$. Once we have a standard pair, its image under the map $\mathcal{F}_0$ is a bunch of unstable curves and some measures living on them.
A nice property of standard pairs is that this image is in fact a weighted sum of standard pairs. That is why we call weighted sums of standard pairs standard families. Formally, a standard family is a set $\mathcal{G} = \{(W_a, \nu_a)\}, a \in \mathfrak{A}$ of standard pairs and a probability measure $\lambda_{\mathcal{G}}$ on the index set $\mathfrak{A}$. This family defines a probability measure on $\mathcal{M}_0$ by

$$\mu_{\mathcal{G}}(B) = \int \nu_a(B \cap W_a) d\lambda_{\mathcal{G}}(a).$$

Every $x \in W_a$ (for some $a \in \mathfrak{A}$), chops W_a into two pieces. The length of the shorter one is denoted by $r_{\mathcal{G}}$. Now the $\mathcal{Z}$-function of $\mathcal{G}$ is defined by

$$\mathcal{Z}_{\mathcal{G}} = \sup_{\varepsilon > 0} \frac{\mu_{\mathcal{G}}(r_{\mathcal{G}} < \varepsilon)}{\varepsilon}.$$

Note that if $\mathcal{G}$ consists of one standard pair, then $\mathcal{Z}_{\mathcal{G}} = 2/|W|$. In any case, we assume $\mathcal{Z}_{\mathcal{G}} < \infty$.

A most important property of the billiard map is that while the unstable curves are expanded due to hyperbolicity, they are also cut by the singularities of $\mathcal{F}_0$; and in some sense, the expansion prevails fragmentation. Formally, the following Growth lemma holds true:

Lemma 1 ([8] Prop 1.). *Let $l = (W, \rho)$ be some standard pair. Then*

$$\mathbb{E}_l(A \circ \mathcal{F}_0^n) = \sum_a c_{a,n} \mathbb{E}_{l_{an}}(A),$$

where $c_{a,n} > 0$, $\sum_a c_{a,n} = 1$; $l_{an} = (W_{an}, \rho_{an})$ are standard pairs such that $\cup_a W_{an} = \mathcal{F}_0^n W$ and ρ_{an} is the pushforward of ρ by $\mathcal{F}_0^n$ up to a multiplicative factor. Finally, there are universal constants $\varkappa, C_1$ (depending only on $\mathcal{D}$), such that if $n > \varkappa |\log length(W)|$, then

$$\sum_{length(l_{an})<\varepsilon} c_{a,n} < C_1 \varepsilon.$$

Another way of stating basically the same lemma is that there are universal constants $\theta < 1, C_2, C_3$ (depending only on $\mathcal{D}$) such that for a standard family $\mathcal{G} = \{(W_a, \nu_a)\}, a \in \mathfrak{A}$, and $\mathcal{G}_n = \mathcal{F}_0^n(\mathcal{G})$, one has

$$\mathcal{Z}_{\mathcal{G}_n} < C_2 \theta^n \mathcal{Z}_{\mathcal{G}} + C_3.$$

If we fix some large constant C_p and call a standard family proper if its $\mathcal{Z}$ function is smaller than C_p, then briefly one can say that the image of $\mathcal{G}$ becomes proper in $\log \mathcal{Z}_{\mathcal{G}}$ steps.

The essence of the standard pair technique is that the measures carried on two proper standard families can be coupled together exponentially fast. Then one of the two standard families is chosen to be μ_0 itself (it can be proven that there exists $\mathcal{G}$ such that $\mu_{\mathcal{G}} = \mu_0$). As a result, one obtains the following Equidistribution statement.

Lemma 2 ([4] Theorem 4). *Let $\mathcal{G}$ be a proper standard family. For any dynamically Hölder continuous f there exists some $\theta_f < 1$ such that for any $n \geq 0$,*

$$\left| \int_{\mathcal{M}_0} f \circ \mathcal{F}_0^n d\mu_{\mathcal{G}} - \int_{\mathcal{M}_0} f d\mu_0 \right| \leq B_f \theta_f^n.$$

We want to identify the possible limit points of the rescaled trajectory of the particle in case of infinite horizon. We assume that there are at least two non-parallel infinite corridors. The observable, we are interested in, is Δ_0. One difficulty is that Δ_0 is not dynamically Hölder continuous if the horizon is infinite. To overcome this problem, we introduce a cut-off of the free flight vector, thus we obtain a dynamically Hölder continuous version. That is, we define $\hat{\Delta}_j$ to be equal to Δ_j once the distance of the scatterers hit by the particle at time j and $j+1$ is less than R, otherwise let $\hat{\Delta}_j$ be zero (we will set $R = \sqrt{N} \log^\beta N$). Note that this is slightly different than $\Delta_j 1_{\{\Delta_j < R\}}$. With this notation, write

$$\hat{q}_j = q_0 + \sum_{i=0}^{j-1} \hat{\Delta}_i.$$

The following lemma is proven in its form in [5] (note also that the first half of part (a) is a purely geometric statement, (b) is a consequence of the Equidistribution, (c) was essentially proven in [12]).

Proposition 1. *Let l be a standard pair.*

(a) $\mathcal{M}_0$ is divided by the singularity curves of $\mathcal{F}_0$ to cells D_m, such that $\Delta_0 \sim Cm$ on D_m. Then for any $n > 0$,

$$\mu_0 \left[D_{m_1} \cap \mathcal{F}^{-n}(D_{m_2}) \right] < min\{C m_2^{-3}, C m_1^{-9/4} m_2^{-2}\}.$$

(b) For any $i \geq 0$,

$$\mathbb{E}_l \left(\hat{\Delta}_i \right) = \mu_0 \left(\hat{\Delta}_i \right) + O\left(\theta^i R \right) = O\left(\theta^i R \right),$$

and analogously, for $k > 1$ and $i_1 \leq i_2 \leq \cdots \leq i_k$,

$$\mathbb{E}_l \left(\hat{\Delta}_{i_1} \otimes \cdots \otimes \hat{\Delta}_{i_k} \right) = \mu_0 \left(\hat{\Delta}_{i_1} \otimes \cdots \otimes \hat{\Delta}_{i_k} \right) + O\left(\theta^{i_1} R^k \right),$$

and

$$\mathbb{E}_l \left(\|\hat{\Delta}_{i_1}\| \dots \|\hat{\Delta}_{i_k}\| \right) = \mu_0 \left(\|\hat{\Delta}_{i_1}\| \dots \|\hat{\Delta}_{i_k}\| \right) + O\left(\theta^{i_1} R^k \right).$$

(c)

$$\mu_0 \left(\hat{\Delta}_j \otimes \hat{\Delta}_j \right) = 2\sigma^2 \log R + O(1),$$

where σ^2 is a non degenerate 2×2-matrix, explicitly given in [12].

(d)

$$\mu_0 \left(\|\hat{\Delta}_j \otimes \hat{\Delta}_k\| \right) < C \theta^{|j-k|},$$

whenever $j \neq k$.

Now, we proceed to our main statement. Let $\mathbf{W}_N(x)$ be element of $C([0,1], \mathbb{R}^2)$ with $\mathbf{W}_N(x)(j/N) = q_j(x)/\sqrt{N \log N}$ and linearly interpolated between j/N and $(j+1)/N$. When x is chosen according to some standard pair l, $\mathbf{W}_N(x)$ generates a measure $\mathbf{W}_{N,l}$ on $C[0,1]$. In this paper, we are going to prove the following statement.

Theorem 1. *Suppose that for some fixed l there is a sequence of integers N_k such that $\mathbf{W}_{N_k,l}$ is weakly convergent. Then its limit is the Wiener measure with covariance matrix σ^2.*

Remark 1. *The statement of Theorem 1 is also true for l being replaced by some proper standard family $\mathcal{G}$. Consequently, for the invariant measure, too.*

3 Proof by martingale method

Here, we are going to prove Theorem 1 by the martingale method of Stroock and Varadhan. That is, we prove the following statement: For any $\mathbf{W}$ limit point of the super-diffusively scaled billiard trajectory,

$$\phi(\mathbf{W}(t)) - \phi(\mathbf{W}(0)) - \frac{1}{2}\int_0^1 \sum_{a,b\in\{1,2\}} D^2_{ab}\phi(\mathbf{W}(s))\sigma^2_{ab}ds$$

is a martingale, where σ^2_{ab} is an element of the super-diffusive covariance matrix, D^2_{ab} stands for partial derivatives of second order, and $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function with compact support. In order to see this, it suffices to prove that for any smooth functions $\psi_1, ...\psi_m$ and for any $0 < s_1 < s_2 < \cdots < s_m < t_1 < t_2$,

$$\mathbb{E}_l\left(\left[\phi(\mathbf{W}(t_2)) - \phi(\mathbf{W}(t_1)) - \frac{1}{2}\int_{t_1}^{t_2} \sum_{a,b\in\{1,2\}} D^2_{ab}\phi(\mathbf{W}(s))\sigma^2_{ab}ds\right]\prod_{j=1}^{k}\psi_j(\mathbf{W}(s_j))\right) = 0. \tag{1}$$

We will prove the following simplified version of (1) (it will be clear, how its proof provides also the more general statement (1)):

$$\mathbb{E}_l\left(\phi(\hat{\mathbf{W}}(t)) - \phi(\hat{\mathbf{W}}(0)) - \frac{1}{2}\int_0^1 \sum_{a,b\in\{1,2\}} D^2_{ab}\phi(\hat{\mathbf{W}}(s))\sigma^2_{ab}ds\right) = 0, \tag{2}$$

(where $\hat{\mathbf{W}}$ is a limiting point of the super diffusively scaled variant of the process $\hat{q}_j$ and l is a standard pair) and

$$\max_{m\le N}\frac{q_m - \sum_{j=0}^{m-1}\hat{\Delta}_j}{\sqrt{N\log N}} \Rightarrow 0, \tag{3}$$

where the weak convergence is with respect to the measure generated by l.
For this fixed $l = (W, \rho)$ and any $x \in W$, $n \geq 0$, define $r_n(x)$ the following way. The image of W under $\mathcal{F}_0^n$ is cut into several homogeneous unstable curves $W_1, W_2, \ldots$. There is an i such that $\mathcal{F}_0^n x \in W_i$. Now, W_i is cut by $\mathcal{F}_0^n x$ into two pieces, the length of the shorter one is denoted by $r_n(x)$. Observe that the growth lemma implies the existence of some C depending on l such that for every $\varepsilon > 0$ and every $n \in \mathbb{Z}_+$,

$$\mathbb{P}_l(r_n(x) < \varepsilon) < C\varepsilon.$$

Important remark: from now on, every appearance of C might mean a different constant. Nevertheless, each C depends only on the length of l (and of course on $\mathcal{D}$). Similarly, O has some involved constant which only depend on $\mathcal{D}$ and the length of l.
Since the free flight of length m is attained by point on $\mathcal{M}$ which belong to some homogeneity strip of width m^{-2}, one obtains

$$\mathbb{P}_l(\hat{\Delta}_n(x) > \sqrt{N}\log^{\beta} N) < \mathbb{P}_l(r_n(x) < N^{-1}\log^{-2\beta} N) < CN^{-1}\log^{-2\beta} N.$$

Thus

$$\mathbb{P}_l(\exists 1 \leq n \leq N : \hat{\Delta}_n \neq \Delta_n) = O(\log^{-2\beta} N) \tag{4}$$

which implies (3).

The rest of this Section is devoted to the proof of (2). Let $\alpha > 0$ be small, $m_p = pN^\alpha$, and fix some $p \geq 2$ integer. For a smooth ϕ with compact support:

$$\begin{aligned}
&\phi\left(\frac{\hat{q}_{m_{p+1}}}{\sqrt{N\log N}}\right) - \phi\left(\frac{\hat{q}_{m_p}}{\sqrt{N\log N}}\right) = \sum_{j=m_p}^{m_{p+1}-1} \phi\left(\frac{\hat{q}_{j+1}}{\sqrt{N\log N}}\right) - \phi\left(\frac{\hat{q}_j}{\sqrt{N\log N}}\right) = \\
= \;& \sum_{j=m_p}^{m_{p+1}-1} \frac{1}{\sqrt{N\log N}} \left\langle D\phi\left(\frac{\hat{q}_j}{\sqrt{N\log N}}\right), \hat{\Delta}_j \right\rangle \\
+ \;& \frac{1}{2}\sum_{j=m_p}^{m_{p+1}-1} \frac{1}{N\log N} \left\langle D^2\phi\left(\frac{\hat{q}_j}{\sqrt{N\log N}}\right)\hat{\Delta}_j, \hat{\Delta}_j \right\rangle + O\left(\frac{\sum_{j=m_p}^{m_{p+1}-1} \|\hat{\Delta}_j\|^3}{(N\log N)^{3/2}}\right) \\
=: \;& S_1^1 + S_2^1 + S_3^1
\end{aligned}$$

Now, using Proposition 1 b, we have

$$\mathbb{E}_l\left(\|\hat{\Delta}_j\|^3\right) < C \sum_{k=1}^{\sqrt{N}\log^\beta N} \frac{1}{k^3}k^3 + O\left(\theta^j N^{3/2}\log^{3/2} N\right) = O\left(N^{1/2}\log^\beta N\right), \tag{5}$$

(here, we also used that due to $p \geq 2$, we have $\theta^j << N$ for any $m_p < j$). Thus we conclude

$$\mathbb{E}_l(S_3^1) = O\left(N^\alpha N^{1/2}\log^\beta N N^{-3/2}\log^{-3/2} N\right) = O\left(N^{\alpha-1}\log^{\beta-3/2} N\right).$$

Further, for $m_p \leq j < m_{p+1}$,

$$\begin{aligned}
D\phi\left(\frac{\hat{q}_j}{\sqrt{N\log N}}\right) &= D\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N\log N}}\right) + \frac{1}{\sqrt{N\log N}} \sum_{k=m_{p-1}}^{j-1} D^2\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N\log N}}\right)\hat{\Delta}_k \\
&+ O\left(\frac{1}{N\log N} D^3\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N\log N}}\right)\left(\hat{q}_j - \hat{q}_{m_{p-1}}\right)^{\otimes 2}\right) =: S_1^2 + S_2^2 + S_3^2,
\end{aligned}$$

where S_3^2 is the error term in the Taylor expansion. Now, we want to substitute $S_1^2 + S_2^2 + S_3^2$ to S_1^1. The substitution of $S_1^2 + S_2^2$ will be computed, while the one of S_3^2 is an error term. To estimate the latter, observe that both coordinates of S_3^2 are bounded by

$$C\frac{1}{N\log N}\left(\sum_{k=m_{p-1}}^{j-1} \|\hat{\Delta}_k\|\right)^2.$$

Thus, when substituting S_3^2 to S_1^1, one obtains a term whose modulus has l-expectation not larger than some constant times

$$\frac{1}{N^{3/2}\log^{3/2}N}\mathbb{E}_l\Big[\sum_{j=m_p}^{m_{p+1}-1}\sum_{m_{p-1}\le k_1\le k_2\le j-1}\|\hat{\Delta}_{k_1}\|\|\hat{\Delta}_{k_2}\|\|\hat{\Delta}_j\|\Big]. \tag{6}$$

To estimate (6), we introduce a second cut-off: let $\hat{\hat{\Delta}}_j$ to be equal to Δ_j once the distance of the scatterers hit by the particle at time j and $j+1$ is less than A_N, otherwise let $\hat{\hat{\Delta}}_j$ be zero (we will set $A_N = N^{9\alpha}$).

Now, we compute the contribution of the Δ's for fixed $k_1 \le k_2 < j$ with (the indices n_1, n_2, n_3 stand for the realization of $\hat{\Delta}_{k_1}$, $\hat{\Delta}_{k_2}$, $\hat{\Delta}_j$, respectively):

$$\begin{aligned}
&\mathbb{E}_l\left[\|\hat{\Delta}_{k_1}-\hat{\hat{\Delta}}_{k_1}\|\|\hat{\Delta}_{k_2}\|\|\hat{\Delta}_j\|\right] < \mu_0\left[\|\hat{\Delta}_{k_1}-\hat{\hat{\Delta}}_{k_1}\|\|\hat{\Delta}_{k_2}\|\|\hat{\Delta}_j\|\right] + O(\theta^{k_1}N^{3/2}\log^{3\beta}N)\\
< \;& C\sqrt{N}\log^\beta N \sum_{n_1=A_N}^{\sqrt{N}\log^\beta N}\sum_{n_3=1}^{\sqrt{N}\log^\beta N} n_1 n_3 n_1^{-9/4}n_3^{-2}\\
< \;& \sqrt{N}\log^\beta N A_N^{-1/4}\log N.
\end{aligned} \tag{7}$$

In the first inequality, we used the Equidistribution (note that $\hat{\Delta}_0-\hat{\hat{\Delta}}_0$ is also dynamically Hölder continuous), while in the second one, we used Proposition 1 (a) and the fact that $p \ge 2$. One can similarly compute that for $k_1 \le k_2 < j$,

$$\begin{aligned}
&\mathbb{E}_l\left[\|\hat{\hat{\Delta}}_{k_1}\|\|\hat{\Delta}_{k_2}\|\|\hat{\Delta}_j\|\right] < A_N\mathbb{E}_l\left[\|\hat{\Delta}_{k_2}\|\|\hat{\Delta}_j\|\right] + O(A_N\theta^{k_2}N\log^{2\beta}N)\\
< \;& CA_N\sum_{n_2=1}^{\sqrt{N}\log^\beta N}\sum_{n_3=1}^{\sqrt{N}\log^\beta N} n_2 n_3 n_2^{-9/4}n_3^{-2} < CA_N\log N.
\end{aligned}$$

Combining the above estimations we conclude that (6) is bounded by

$$\begin{aligned}
&CN^{-3/2}\log^{-3/2}N\left(N^{3\alpha}\cdot\sqrt{N}\log^{\beta+1}NA_N^{-1/4} + N^{3\alpha}\cdot A_N\log N\right)\\
= \;& o(N^{\alpha-1}),
\end{aligned}$$

if we choose $A_N = N^{9\alpha}$, and α is small. At the last step, we want to substitute

$$D^2\phi\left(\frac{\hat{q}_j}{\sqrt{N\log N}}\right) \tag{8}$$

in S_2^1 with

$$D^2\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N\log N}}\right). \tag{9}$$

It is easy to see that the difference between S_2^1 and the formula obtained from S_2^1 with (8) being replaced by (9) is in

$$O\left(\frac{1}{(N\log N)^{3/2}}\sum_{m_p\le j\le m_{p+1}-1}\ \sum_{m_{p-1}\le k<j}\|\hat{\Delta}_k\|\|\hat{\Delta}_j\|^2\right) \tag{10}$$

As before, we have

$$\mathbb{E}_l\Big[\|\hat{\hat{\Delta}}_k\|\|\hat{\Delta}_j\|^2\Big]\le CA_N\sum_{n_2=1}^{\sqrt{N}\log^{\beta}N} n_2^{-3}n_2^2+O\left(A_N\theta^jN\log^{2\beta}N\right)<CA_N\log N. \tag{11}$$

This, and (7) imply that the expectation of (10) with respect to l is bounded by

$$CN^{-3/2}\log^{-3/2}N\cdot N^{2\alpha}\Big(\sqrt{N}\log^{\beta+1}NA_N^{-1/4}+A_N\log N\Big)=o(N^{\alpha-1}).$$

Hence, for $p\ge 2$ we obtain

$$\begin{aligned}
&\phi\left(\frac{\hat{q}_{m_{p+1}}}{\sqrt{N\log N}}\right)-\phi\left(\frac{\hat{q}_{m_p}}{\sqrt{N\log N}}\right)\\
=&\sum_{j=m_p}^{m_{p+1}-1}\frac{1}{\sqrt{N\log N}}\Big\langle D\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N\log N}}\right),\hat{\Delta}_j\Big\rangle\\
+&\frac{1}{N\log N}\Big[\frac{1}{2}\sum_{j=m_p}^{m_{p+1}-1}\Big\langle D^2\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N\log N}}\right)\hat{\Delta}_j,\hat{\Delta}_j\Big\rangle\\
+&\sum_{m_p\le j<m_{p+1},m_{p-1}\le k<j}\Big\langle D^2\phi\left(\frac{\hat{q}_{m_{p-1}}}{\sqrt{N\log N}}\right)\hat{\Delta}_k,\hat{\Delta}_j\Big\rangle\Big]+h_p,
\end{aligned} \tag{12}$$

where $\mathbb{E}_l(h_p)=o\left(N^{\alpha-1}\right)$.
With the notation

$$f_p(x)=\phi\left(\frac{\hat{q}_{m_{p+1}}(x)}{\sqrt{N\log N}}\right)-\phi\left(\frac{\hat{q}_{m_p}(x)}{\sqrt{N\log N}}\right),$$

for any $x\in\mathcal{M}$,

$$\phi\left(\frac{\hat{q}_N(x)}{\sqrt{N\log N}}\right)-\phi\left(\frac{\hat{q}_0(x)}{\sqrt{N\log N}}\right)=\sum_{p=0}^{N^{1-\alpha}}f_p(x)$$

Thus, in order to verify (2), we need to prove

$$\mathbb{E}_l\sum_{p=0}^{N^{1-\alpha}}\Big[f_p(x)-N^{\alpha-1}\frac{1}{2}\sum_{a,b\in\{1,2\}}D^2_{ab}\phi\left(\frac{\hat{q}_{m_p}(x)}{\sqrt{N\log N}}\right)\sigma^2_{ab}\Big]=o(1). \tag{13}$$

First, we verify that $\mathbb{E}_l(f_0+f_1)=o(1)$. Note that Proposition 1 (b) implies

$$\mathbb{E}_l\|\hat{q}_{m_2}-\hat{q}_0\| < \sum_{k=0}^{m_2}\mu_0(\|\hat{\Delta}_k\|)+O(\theta^k\sqrt{N}\log^\beta N)=O(\sqrt{N}\log^\beta N).$$

Since $\beta<1/2$, the Markov inequality and the fact that ϕ has compact support and is in C^1, implies $\mathbb{E}_l(f_0+f_1)=o(1)$.
For $p\geq 2$, as in [8], we want to use the Markov decomposition at time $\tau_p=(m_{p-1}+m_p)/2$. Now, we will also need some Markov decomposition at time $\tilde{\tau}_p=(3m_{p-1}+m_p)/4$. However, observe that since $\hat{q}_{m_p}$ is not necessarily equal to q_{m_p}, $f_p(x)$ and

$$f'_p(x)=\phi\left(\frac{\hat{q}_{\frac{3}{2}N^\alpha}(\mathcal{F}^{m_{p-1}+N^\alpha/2}x)}{\sqrt{N\log N}}\right)-\phi\left(\frac{\hat{q}_{\frac{1}{2}N^\alpha}(\mathcal{F}^{m_{p-1}+N^\alpha/2}x)}{\sqrt{N\log N}}\right),$$

which is easier to deal with using the Markov decomposition, are not equal in general. That is why we need some more computation. Observe that the decomposition (12) can also be written in terms of f'_p instead of f_p. The difference is that $\hat{q}_{m_{p-1}}$ should be replaced by

$$\hat{q}'_{m_{p-1}}=q_{(m_{p-1}+m_p)/2}-\sum_{j=1}^{N^\alpha/2}\hat{\Delta}_j,$$

and h_p should be replaced by some h'_p. Observe that our previous computation also yields $\mathbb{E}_l(h'_p)=o(N^{\alpha-1})$.

Now, we claim that

$$\lim_{N\to\infty}\sum_{p=2}^{N^{1-\alpha}}\mathbb{E}_l\left(f_p-f'_p\right)=0. \tag{14}$$

To prove (14), observe that with the notation

$$\mathcal{L}=\{x\in\mathcal{M}:\exists j\leq N:\Delta_j(x)\neq\hat{\Delta}_j(x)\}$$

for the set of points having long flight, f_p coincides with f'_p on $\mathcal{M}\setminus\mathcal{L}$, which has l-measure at least $1-C\log^{-2\beta}N$ by (4). Since $\sum_p f_p$ is bounded,

$$\int_{\mathcal{L}}\sum_p f_p dl=o(1).$$

Thus in order to prove (14), is is enough to establish

$$\int_{\mathcal{L}}\sum_p f'_p dl=o(1). \tag{15}$$

This statement is not obvious, since $\sum_p f'_p$ is not bounded. However, with the notation

$$L(x) = \#\{p < N^{1-\alpha} : \exists j \in [m_p, m_{p+1}], \Delta_j(x) \neq \hat{\Delta}_j(x)\},$$

we have for any $x \in \mathcal{L}$,

$$\sum_p f'_p(x) < 2(L(x)+1)\|\phi\|.$$

Thus, it is enough to prove

$$\int_{\mathcal{L}} L(x) dl = o(1). \tag{16}$$

We will prove that

$$\int_{\mathcal{L}} L_{even}(x) dl = o(1), \tag{17}$$

where

$$L_{even}(x) = \#\{p < N^{1-\alpha}, p \text{ is even and } \exists j \in [m_p, m_{p+1}] \Delta_j(x) \neq \hat{\Delta}_j(x)\}.$$

This, together with a very same computation for the odd p's implies (16). To prove (17), first observe that

$$\mathbb{P}_l(L_{even}(x) = 1) < C \log^{-2\beta} N.$$

We claim that analogously,

$$\mathbb{P}_l(L_{even}(x) = k) < C^k \log^{-2\beta k} N \tag{18}$$

for every k positive integer, with a uniform C. This implies (17), since $\sum_{k>1} kC^k \log^{-2\beta k} N = o(1)$. Now pick any $1 \leq i < i + N^\alpha \leq j \leq N$. We prove that

$$\mathbb{P}_l(\Delta_i \neq \hat{\Delta}_i, \Delta_j \neq \hat{\Delta}_j) < C^2 N^{-2} \log^{-4\beta} N, \tag{19}$$

which obviously implies (18) for $k = 2$ (for larger k's, the proof goes the same way). We have

$$\mathbb{P}_l(\Delta_i \neq \hat{\Delta}_i, \Delta_j \neq \hat{\Delta}_j) = \sum_a c_a \mathbb{P}_{l_a}(\Delta_{j-i} \neq \hat{\Delta}_{j-i}), \tag{20}$$

where $\{l_a\}_a$ is the collection of standard pairs in the image of l under $\mathcal{F}^{i+1}$, for which for any point x in γ_a, $\Delta_0(\mathcal{F}^{-1}x) \neq \hat{\Delta}_0(\mathcal{F}^{-1}x)$. We already know that $\sum_a c_a < CN^{-1} \log^{-2\beta} N$. Let $S_1 + S_2$ be the sum in (20), where S_1 corresponds to a's, for which $length(l_a) < N^{-3}$. Then the growth lemma implies $S_1 < CN^{-3}$. For a's, where $length(l_a) > N^{-3}$, the image of l_a becomes proper in $C \log N$ steps, thus (the proof of (4)) implies $S_2 < \sum_a c_a CN^{-1} \log^{-2\beta} N$ with a uniform C. Thus we have verified (19), and finished the proof of (14).

As it was already mentioned, we will also need a Markov decomposition at time $(3m_{p-1}+m_p)/4$. Thus, we still need to slightly adjust f_p, that is to define

$$\hat{q}''_{m_{p-1}} = q_{(3m_{p-1}+m_p)/4} - \sum_{j=1}^{N^\alpha/4} \hat{\Delta}_j,$$

and

$$\begin{aligned} f''_p(x) \quad := & \quad \sum_{j=m_p}^{m_{p+1}-1} \frac{1}{\sqrt{N\log N}} \Big\langle D\phi\left(\frac{\hat{q}'_{m_{p-1}}}{\sqrt{N\log N}}\right), \hat{\Delta}_j \Big\rangle \\ & + \quad \frac{1}{N\log N}\Big[\frac{1}{2}\sum_{j=m_p}^{m_{p+1}-1} \Big\langle D^2\phi\left(\frac{\hat{q}'_{m_{p-1}}}{\sqrt{N\log N}}\right)\hat{\Delta}_j, \hat{\Delta}_j\Big\rangle \\ & + \quad \sum_{m_p\le j<m_{p+1}, m_{p-1}\le k\le m_{p-1}+\frac{3}{8}N^\alpha} \Big\langle D^2\phi\left(\frac{\hat{q}'_{m_{p-1}}}{\sqrt{N\log N}}\right)\hat{\Delta}_k, \hat{\Delta}_j\Big\rangle \\ & + \quad \sum_{m_p\le j<m_{p+1}, m_{p-1}+\frac{3}{8}N^\alpha<k<j} \Big\langle D^2\phi\left(\frac{\hat{q}''_{m_{p-1}}}{\sqrt{N\log N}}\right)\hat{\Delta}_k, \hat{\Delta}_j\Big\rangle\Big] + h'_p. \end{aligned} \tag{21}$$

Next, we prove that the contribution of this adjustment asymptotically vanishes, i.e.

$$\mathbb{E}_l(|f'_p - f''_p|) = o\left(N^{\alpha-1}\right). \tag{22}$$

In order to prove (22), write

$$\mathcal{L}_p = \{x \in \mathcal{M} : \exists j \in [m_{p-1}+\frac{1}{4}N^\alpha, m_{p-1}+\frac{1}{2}N^\alpha] : \Delta_j(x) \ne \hat{\Delta}_j(x)\}$$

for the set of phase points, where $f'_p \ne f''_p$. Further, observe that for any $x \in \mathcal{L}_p$, the only difference between $f'_p(x)$ and $f''_p(x)$ is that in the fourth line of (21) $\hat{q}''_{m_{p-1}}$ is replaced by $\hat{q}'_{m_{p-1}}$ in the case of $f'_p(x)$. Thus, for any $x \in \mathcal{L}_p$,

$$|f'_p - f''_p| < C\frac{1}{N\log N}N^\alpha\sqrt{N}\log^\beta N \sum_{m_p\le j<m_{p+1}} \|\hat{\Delta}_j\|.$$

Consequently, the Markov decomposition at time $m_{p-1}+N^\alpha/2$ implies

$$\int_{\mathcal{L}_p} |f'_p - f''_p| dl < CN^{-1+\alpha+1/2}\sum_a c_a \sum_{m_p\le j<m_{p+1}} \mathbb{E}_{l_a}(\|\hat{\Delta}_j\|), \tag{23}$$

where $\{l_a\}_a$ is the collection of standard pairs in the image of l under $\mathcal{F}^{m_{p-1}+N^\alpha/2}$, for which for any point x in γ_a, there is a $j \in [0, N^\alpha/4]$, such that $\hat{\Delta}_0(\mathcal{F}^{-j}x) \ne \Delta_0(\mathcal{F}^{-j}x)$. Note that $\sum_a c_a = l(\mathcal{L}_p)$. If we denote by S_1+S_2 the sum in (23), where S_1 corresponds to a's with $length(l_a) < N^{-2}$, then using the obvious estimation $\mathbb{E}_{l_a}(\|\hat{\Delta}_j\|) < \sqrt{N}\log^\beta N$ and the growth lemma, we obtain $S_1 < CN^{2\alpha-2}\log^\beta N$. Since in $C\log N$ steps the standard pairs of the sum S_2 develop to proper families, Proposition 1 (b) and (4) imply $S_2 < N^{-1+\alpha+1/2}l(\mathcal{L}_p)N^\alpha < N^{-1+\alpha+1/2+\alpha-1+\alpha}$. (22) follows.

Combining (14) and (22), (13) is equivalent to the statement

$$\mathbb{E}_l\left(f_p''\right) = N^{\alpha-1}\frac{1}{2}\sum_{a,b\in\{1,2\}}\mathbb{E}_l\left(D^2_{ab}\phi\left(\frac{\hat{q}_{m_p}(x)}{\sqrt{N\log N}}\right)\right)\sigma^2_{ab}(1+o(1)). \tag{24}$$

Since $\mathbb{E}_l(h_p')$ is in $o(N^{\alpha-1})$, in order to prove (24), it suffices to verify that $T_1+T_2+T_3+T_4$ is equal to the right hand side of (24), where T_i is the l-expected value of line i in formula (21) - except for T_4, where we omit h_p'. The proof of this is similar to the one in [8].
So as to estimate T_1, T_2 and T_3, we use Markov decomposition at time $(m_{p-1}+m_p)/2$. Since in any case, the first three lines of (21) are bounded by $CN^{2\alpha}$, the standard pairs that are shorter than N^{-2} contribute to $T_1+T_2+T_3$ with a term which is bounded by $CN^{-2+2\alpha}$. If we denote by T_1', T_2' and T_3' the contribution of the standard pairs that are longer than N^{-2}, then we have

$$T_1' = O\left(\frac{1}{\sqrt{N\log N}}N^\alpha\theta^{\frac{1}{2}N^\alpha}\sqrt{N}\log^\beta N\right). \tag{25}$$

Indeed, the value of $D\phi\left(\frac{\hat{q}'_{m_{p-1}}}{\sqrt{N\log N}}\right)$ on some points which will form a standard pair at time $(m_{p-1}+m_p)/2$ is some constant with error $O(\theta^{\frac{1}{2}N^\alpha})$, thus Proposition 1 (b) implies (25).
Now, using Proposition 1 (c) one can analogously compute that

$$T_2' = N^{\alpha-1}\frac{1}{2}\sum_{a,b\in\{1,2\}}\mathbb{E}_l\left(D^2_{ab}\phi\left(\frac{\hat{q}_{m_p}(x)}{\sqrt{N\log N}}\right)\right)\sigma^2_{ab}(1+o(1)). \tag{26}$$

Similarly to the estimation of T_1', we can bound T_3'. Note that $D^2\phi\left(\frac{\hat{q}'_{m_{p-1}}}{\sqrt{N\log N}}\right)\hat{\Delta}_k$ on some points which will form a standard pair at time $(m_{p-1}+m_p)/2$ is some constant with error $O(\theta^{\frac{1}{2}N^\alpha})$, but now, this constant is only bounded by $\sqrt{N}\log^\beta N$. Thus again, Proposition 1 (b) yields

$$T_3' = O\left(\frac{1}{N\log N}N^{2\alpha}\sqrt{N}\log^\beta N\theta^{\frac{1}{8}N^\alpha}\sqrt{N}\log^\beta N\right). \tag{27}$$

Finally, we use Markov decomposition at time $(3m_p+m_{p+1})/4$ to estimate T_4. As before, since the last line of (21) is bounded by $CN^{2\alpha}$, the standard pairs that are shorter than N^{-2} can be neglected. Using the same argument as in the proof of (25), and now also

Proposition 1 (d), we conclude that the contribution of the longer standard pairs is in

$$O\left(\frac{1}{N\log N}\sum_{m_p\le j<m_{p+1},m_{p-1}+\frac{3}{8}N^\alpha<k<j}\left(\theta^{j-k}+\theta^{\frac{1}{8}N^\alpha}N\log^{2\beta}N\right)\right).$$

This, together with our previous estimations, yields (24). We have finished the proof of (2).

References

[1] Bleher, P. M.: Statistical Properties of Two-Dimensional Periodic Lorentz Gas with Infinite Horizon, Journal of Stat. Phys. 66 (1): 315-373 (1992)

[2] Bunimovich, L. A., Sinai, Ya. G., "Statistical properties of Lorentz gas with periodic configuration of scatterers". Commun. Math. Phys. 78:479-497 (1981).

[3] Bunimovich, L. A., Sinai, Ya. G., Chernov, N., "Statistical properties of two-dimensional hyperbolic billiards", Russ. Math. Surv. 46:47-106 (1991).

[4] Chernov, N.: Advanced Statistical Properties of Dispersing Billiards, Journal of Stat. Phys. 122:1061-1094 (2006).

[5] Chernov, N., Dolgopyat, D.: Anomalous current in periodic Lorentz gases with infinite horizon, Russian Mathematical Surveys, 64, 651-699 (2009).

[6] Chernov, N., Dolgopyat. D.: Brownian Brownian Motion–1. Memoirs of American Mathematical Society 198 no. 927 (2009).

[7] Chernov, N., Markarian, R.: Chaotic billiards Mathematical Surveys and Monographs, 127, AMS, Providence, RI (2006).

[8] Dolgopyat, D., Szász, D., Varjú, T.: Limit Theorems for Locally Perturbed Lorentz Process, Duke Mathematical Journal, 148 (2009).

[9] Lorentz, H.: Le mouvement des électrons dans les métaux. Arch. Néerl. 10 336–371, (1905).

[10] Sinai, Ya.: Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards. Russian Math. Surv. 25 137-189 (1970).

[11] Stroock, D. W., Varadhan, S. R. S.: Multidimensional Diffusion Processes, Classics in Mathematics, Springer-Verlag, Berlin (2006).

[12] Szász, D., Varjú, T.: Limit Laws and Recurrence for the Planar Lorentz Process with Infinite Horizon, Journal of Stat. Phys. 129:59-80 (2007).