Dynamical systems, Spring 2024

Homework problem set #1 . Due on March 26, Tuesday

One of the 11 problems below can be regarded as a bonus problem. That is, with complete solutions for 10 problems you can obtain full credit. Solving all the 11 problems properly deserves extra credit.

1. Consider the one dimensional map $T_\lambda : \mathbb{R} \to \mathbb{R}$, $T_\lambda x = x^5 - \lambda x$, where the parameter λ satisfies $-\infty < \lambda \leq 1$. Investigate the λ-dependence of
   (a) the fixed points and their stability properties.
   (b) the asymptotic behavior of the orbit $T_\lambda^n x$, $n \geq 0$ for any initial condition $x \in \mathbb{R}$.

2. Consider the doubling map $T : \mathbb{S}^1 \to \mathbb{S}^1$, $Tx = 2x \pmod 1$. Let $D \subset \mathbb{S}^1$ denote the set of points x such that $\{T^n x | n = 0, 1, 2 \dots\}$ is dense in $\mathbb{S}^1$. Prove that $\lambda(D) = 1$ (where λ is the Lebesgue measure).

3. We mentioned in class that

$$T : [0,1] \to [0,1], \qquad T(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1/x \pmod 1 & \text{if } x \neq 0. \end{cases}$$

   (the Gauss map) has an absolutely continuous invariant (probability) measure, with density $\rho(x) = \frac{1}{\ln 2} \frac{1}{1+x}$. Provide a detailed argument for this fact.

4. Recall from class that on $\Sigma^+ = \{0,1\}^{\mathbb{N}}$, that is, the space of infinite sequences of 2 symbols, we defined a metric as $d(\underline{a}, \underline{b}) = 2^{-s(\underline{a},\underline{b})}$, where $s(\underline{a}, \underline{b}) = \min\{k \geq 1 | a_k \neq b_k\}$ (here $\underline{a} = (a_1, a_2, ...)$ and similarly for $\underline{b}$). Prove that $d(\underline{a}, \underline{b})$ is indeed a metric, in particular, it satisfies the triangular inequality.

5. Consider $T : [0,1] \to [0,1]$, $Tx = 4x(1-x)$ (the logistic map with $\mu = 4$) . Verify that T has an absolutely continuous invariant (probability) measure, with density $\rho(x) = C(x(1-x))^{-1/2}$. ($C =$?)

6. Fix a positive integer $K \geq 1$, and let $\Sigma_K^+ = \{0, 1, ..., K-1\}^{\mathbb{N}}$, that is, the space of infinite sequences of K symbols. The shift map $\sigma : \Sigma_K^+ \to \Sigma_K^+$ is defined in the usual way.
   (a) What are the periodic points of this shift map with K symbols? Specify a point in Σ_K^+ that has a dense orbit.
   (b) Consider now the map $T_K : \mathbb{S}^1 \to \mathbb{S}^1$, $T_K x = Kx \pmod 1$. How are these two dynamical systems related?

7. Show that the map $T : [-1,1] \to [-1,1]$, $Tx = 8x^4 - 8x^2 + 1$ has an absolutely continuous invariant (probability) measure, and determine the density. (*Hint*: in class we discussed the case of $Tx = 2x^2 - 1$, you may proceed along the same lines, just instead of $2x \pmod 1$ consider $4x \pmod 1$ as a map of the unit circle in $\mathbb{C}$.)

8. Consider the linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) = Ax$ for the matrices A below. In each case, describe the asymptotic behavior and sketch the phase portrait. In hyperbolic cases, determine the stable and unstable subspaces (W^s and W^u).

   (a) $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 \\ 0 & 1/3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1/5 \\ 1/5 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$.

9. Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be a hyperbolic toral automorphism. Show that, for the matrix A associated to T, the eigenvalues are irrational numbers while the eigendirections, as lines on $\mathbb{R}^2$, have irrational slope.

10. Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be a hyperbolic toral automorphism. Show that $x \in \mathbb{T}^2$ is a periodic point for T if and only if both of its coordinates are rational.

11. Consider the logistic map $T_\mu x = \mu x(1-x)$ with $\mu > 2+\sqrt{5}$. Show that there exists some $\lambda > 1$ such that $|T'_\mu(x)| > \lambda$ whenever $x \in I_0 \cup I_1$. (Recall that we denoted $T_\mu^{-1}[0,1] = I_0 \cup I_1$, where I_0 and I_1 are disjoint intervals.)