Dynamical systems, Spring 2024

Homework problem set #3. Due on May 21, Tuesday

One of the 8 problems below can be regarded as a bonus problem. That is, with complete solutions for 7 problems you can obtain full credit. Solving all the 8 problems properly deserves extra credit.

1. We discussed in class that $\Lambda \subset M$ is an *attractor* for the invertible topological dynamical system $T : M \to M$ if there exists an open neighborhood $U \supset \Lambda$ such that for the closure $N = \overline{U}$ it holds that $T(N) \subset U$ (that is, U is a *trapping region*) and $\Lambda = \bigcap_{n=0}^{\infty} T^n N$. Show that in such a case Λ is a closed invariant set (invariance means $\Lambda = T(\Lambda) = T^{-1}(\Lambda)$).

2. As usual, let us represent $\mathbb{T}^2$ as $[0,1]^2$ with the opposite sides identified. Let $a > 0$ be some small parameter (say $a < \frac{1}{100}$) and consider $F : \mathbb{T}^2 \to \mathbb{T}^2$, $F(x,y) = (x + a\sin(2\pi x), y + a\cos(2\pi x)\sin(2\pi y))$. Identify the fixed points, verify that all of them are hyperbolic, and classify them into sources, sinks and saddles. Find the (global) stable and unstable manifolds for each fixed point.

3. Consider the one-sided full shift with two symbols, $\sigma : \Sigma^+ \to \Sigma^+$. Prove that this system has the shadowing property. That is, given a δ-pseudo orbit construct a point the true orbit of which is ε-shadowing the pseudo orbit (where δ is appropriately chosen for ε).

4. Let M be a compact metric space, and for $\varepsilon > 0$ let $C(\varepsilon)$, $N(\varepsilon)$ and $S(\varepsilon)$ denote the minimum cardinality of ε-covers, the minimum cardinality of ε-nets, and the maximum cardinality of ε-separated sets, respectively. Prove that $C(2\varepsilon) \le N(\varepsilon) \le S(\varepsilon) \le C(\varepsilon)$.

5. Prove (ii), (iii), (iv), (vi), (vii) and (viii) from the list of properties of $H(\alpha)$ and $H(\alpha|\beta)$. (You may rely on (i) and (v).)

6. Prove (1), (2), (3), (5), (6) and (7) from the list of properties of $h(T)$. (You may rely on (4) and (i-xii).)

7. Let A be a primitive adjacency matrix (i.e. the entries A_{kl} take values 0 or 1, and $\exists N \ge 1$ such that $(A^N)_{kl} > 0$ for any k, l). Then according to the Perron-Frobenius theorem A has a simple largest eigenvalue, to be denoted by $\lambda > 0$. Let furthermore u_k and s_k denote the associated left and right eigenvectors, respectively, normalized so that $\sum_{k=0}^{K-1} s_k u_k = 1$.
   (a) Verify that $\pi_{kl} = \lambda^{-1} u_k^{-1} A_{kl} u_l$ is the transition matrix of an irreducible aperiodic Markov chain, and that the corresponding stationary distribution is $p_k = s_k u_k$.
   (b) Consider the associated Markov shift, compute its entropy and conclude that this is a measure of maximal entropy.

8. (a) Let us fix the parameters $d_1, ..., d_r \in \mathbb{R}$, and introduce the notation $Z = \sum_{i=1}^{r} e^{d_i}$. Consider the simplex
$$\Delta = \{ \underline{p} = (p_1, ..., p_r) \in \mathbb{R}^r \mid p_i \ge 0, \sum_{i=1}^{r} p_i = 1 \},$$
and the function $F : \Delta \to \mathbb{R}$, $F(\underline{p}) = -\sum_{i=1}^{r} p_i \log p_i + \sum_{i=1}^{r} d_i \cdot p_i$. Show that the maximum of $F(\underline{p})$ on Δ is $\log Z$, attained at the unique point $p_j = \frac{e^{d_j}}{Z}$, $j = 1, ..., r$.

(b) Let us introduce furthermore

$$\Delta_s = \{ \ \underline{p} = (p_1, ..., p_r) \in \mathbb{R}^r \mid p_i \geq 0, \ \sum_{i=1}^{r} p_i = s \ \},$$

for $0 < s \leq 1$. Show that the maximum of F on Δ_s is $s(\log Z - \log s)$, taken at the point $p_j = \frac{se^{d_j}}{Z}$, $j = 1, ..., r$.