Dynamical systems, Spring 2024

Log: a brief summary of the classes

February 13

Dynamical system in discrete time: map. Orbit, periodicity, asymptotic behavior. Invertibility.

Further classification and connections depending on the preserved structure: topological dynamics, ergodic theory, smooth dynamics.

Dynamics in continuous time: flows, autonomous ODE systems.

Rotations of the circle. $\mathbb{S}^1$ as phase space. Invertibility, isometry, rigidity. Rational α: every point is periodic with the same period. Irrational α: every point has a dense orbit (definitions of topological transitivity and minimality) Lebesgue measure is invariant. Further invariant measures concentrated on periodic orbits in the rational case. Outlook: unique ergodicity for irrational α.

February 19

Doubling map or $2x \pmod 1$. Non-invertibility, expansion, dyadic rationals are eventually fixed.

One sided full shift with two symbols. $\Sigma^+ = \{0,1\}^{\mathbb{N}}$ as a compact space, metric, $\sigma : \Sigma^+ \to \Sigma^+$, the left shift.

Equivalence of dynamical systems. conjugacy. Further issues: continuity, measurability, push forward of a measure.

The doubling map and the one sided full shift are (almost) conjugate, discussion of the conjugacy. Applications: characterization of periodic points and points with dense orbits. Invariance of Lebesgue measure for the doubling map. Cylinder sets, description of the $1/2 - 1/2$ Bernoulli measure, when pushed forward, gives Lebesgue.

Further comments on the topology of the shift space: tryadic Cantor set.

Many further invariant measures for the shift.

February 20

Semi-conjugacy, factors.

Doubling map as $Tz = z^2$ on the complex unit circle. How $f : [-1,1] \to [-1,1]$, $f(x) = 2x^2 - 1$ is obtained as a factor, invariant density for the later. (Reminder: density transformation formula.)

Products of dynamical systems: products of two rotations as a map of the torus $\mathbb{T}^2$.

Linear flow on $\mathbb{T}^2$.

Relating flows to maps and back: suspension flow and Poincaré section. Linear flow on the torus as a suspension of a rotation.

Linear self-maps of the real line. Continuous maps of the real line: graphical analysis (cobweb plot), attracting and repelling fixed points.

The logistic family $T_\mu x = \mu x(1-x)$, motivation from population dynamics. Fixed points: the case $\mu \le 1$. $\mu > 1$: orbits of $x \notin [0,1]$.

February 26

Analysis reminder: intermediate value theorem, mean value theorem, implicit function theorem.

A fixed point cannot disappear or split unless $f'(x_0) = 1$.

Logistic family: description of the attracting fixed point for $1 < \mu \le 2$ and $2 < \mu < 3$.

$\mu = 3$: discussion of the second iterate, inflection point, period-doubling bifurcation.

Illustration for the complexity of $3 < \mu < 4$.

Discussion of logistic maps with $\mu > 4$ (for simplicity restrict to $\mu > 2 + \sqrt{5}$). Intervals I_0 and I_1, inverse branches, construction of the invariant Cantor set. Topological conjugacy with the one-sided shift. Repeller.

February 28

Periodic points for continuous maps $T : \mathbb{R} \to \mathbb{R}$. Existence of a period 3 orbit implies existence of periodic orbits with (least) period n for every $n \in \mathbb{N}$. Statement of Sharkovsky's theorem (without proof).

Saddle-node bifurcation in one dimension.

C^r metrics. Structural stability. $\frac{1}{2}x$ is C^1-structurally stable, logistic map with $\mu > 2 + \sqrt{5}$ is C^2-structurally stable.

March 4

Gauss map. Connection to continued fraction expansions. Rational points are eventually fixed. The golden mean as a fixed point of the Gauss map.

Perron-Frobenius operator. Invariant density for the Gauss map.

Linear maps of the plane. Reminder: Jordan canonical form for real matrices.

How the phase portrait is determined by the spectrum: source, sink, saddle, focus, node. Verifying stability by Lyapunov functions. Stable and unstable subspaces.

March 5

Two dimensional nonlinear maps: behavior in the vicinity of a fixed point. Hyperbolic fixed points: Hartman-Grobman theorem. Hyperbolicity of the fixed point is essential: examples of non-hyperbolic fixed points with attracting/repelling behavior. Hopf bifurcation.

Toral automorphisms, definitions, invertibility. Discussion of an elliptic ($\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$) and a parabolic ($\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$) example.

Hyperbolic toral automorphisms, discussed via the particular example of the CAT map $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$: sketch of domains, description of periodic points. Stable and unstable manifolds and foliations. Backward and forward asymptotic points, consequences of the irrational slope. Existence of a dense set of points homoclinic to the origin.

March 6

Two equivalent characterizations of topological transitivity (Baire category theorem).

Hyperbolic toral automorphisms are topologically transitive; proof based on the density of homoclinic points.

Smale's horseshoe map. The sets H_0, H_1, V_0, V_1, $H_{00}, H_{10}, H_{11}, H_{01}$ etc. $\Lambda = \Lambda^+ \cap \Lambda^- = \Lambda_1 \times \Lambda_2$, as the product of two Cantor sets.

Double sided full shift: product topology, separation metric, invertibility. Topological conjugacy with the Smale horseshoe. Periodic points, stable and unstable sets of a point, homoclinic points – geometric and symbolic characterization.

March 11

Ergodic theory. Introductory examples: three piecewise linear maps with constant slope 2. Scope of ergodic theory: invariant measures and their properties. Invariant sets and invariant functions, corresponding equivalent definitions of ergodicity.

$T : M \to M$ with invariant measure μ, associated linear isometry $\hat{T}$ on the Banach spaces L^p_μ. The case of the Hilbert space L^2_μ; definition of $\hat{T}^*$, $f = \hat{T}^* f \Leftrightarrow f = \hat{T} f$.

Birkhoff's and von Neumann's ergodic theorems. Proof of von Neumann's ergodic theorem.

Discussion of the ergodic case: strong law of large numbers. Review of the introductory examples from the viewpoint of ergodicity.

March 12

X compact metric space: $C(X)$ Banach space of continuous functions (sup norm), $\mathcal{M}$ collections of Borel measures on X, as a subset of $C^*(X)$.

Borel measures as positive bounded linear functionals, Riesz representation theorem. Weak-$*$ topology, Banach-Alaoglu theorem.

$T : X \to X$ continuous map, associated operators: $\hat{T} : C(X) \to C(X)$ pull back of functions and $T_* : \mathcal{M} \to \mathcal{M}$ push forward of measures. T_* is continuous in the weak-$*$ topology. $\mathcal{M}_{\text{inv}}$ as the fixed point set of T_*, closed and convex.

Krylov-Bogolyubov theorem, proof based on the ergodic averages of Dirac measures, discussion, alternative proof based on the Schauder fixed point theorem.

Example: $T : [0,1] \to [0,1]$, $Tx = x/2$, $\mathcal{M}_{\text{inv}} = \{\delta_0\}$. Outlook: $\mathcal{M}_{\text{inv}}$ for irrational rotations (unique ergodicity) and the doubling map ($\mathcal{M}_{\text{inv}}$ is large).

Extreme points of a convex set. $\mu \in \mathcal{M}_{\text{erg}}$ if and only if it is an extreme point of $\mathcal{M}_{\text{inv}}$. As a byproduct: if $\mu \in \mathcal{M}_{\text{erg}}$, $\mu_1 \in \mathcal{M}_{\text{inv}}$, $\mu_1 \ll \mu$, then $\mu_1 = \mu$.

If $m, \mu \in \mathcal{M}_{\text{erg}}$, either $m = \mu$ or $m \perp \mu$.

March 19

von Neumann's ergodic theorem (with proof). Birkhoff's ergodic theorem (without proof).

Equivalent characetrizations of ergodicty by the averaged correlations of sets and functions.

Definition of mixing via sets and functions, analogy: decay of correlations. Mixing implies ergodicity.

March 25

Irrational rotations: ergodicity, unique ergodicity, lack of mixing. Weyl's theorem. Arnold's problem on the frequency of the first decimal digits in the sequence 2^n.

Reminder: one sided and double sided full shifts, cylinder sets.

Bernoulli shift.

March 26

Mixing of Bernoulli shifts.

Recap of finite Markov chains. Transition probabilities, stochastic matrices, adjacency matrix. Irreducible and primitive (irreducible aperiodic) cases. Stationary distribution.

April 8

Perron's theorem, spectral gap, exponential convergence to the stationary distribution (in the primitive case).

Adjacency matrix. Topological Markov chains: phase space as a compact invariant subspace of the full shift. Outlook: subshifts of finite type.

Markov shifts. Measure of cylinder sets via the stationary distribution and the transition probabilities.

April 9

ω-limit sets, periodic, recurrent and non-wandering points.

Banach space of Hölder continuous functions. Definition of Exponential decay of correlations. Conditional expectations w. r. to the σ-algebra generated by cylinder sets of length ℓ. Exponential decay of correlations for mixing Markov shifts.

April 15

Recap of hyperbolic toral automorphisms, in particular the CAT map. Rectangles, s- and u-subrectangles, proper intersection. Markov partitions.

Significance of Markov partitions: associated transition probabilities, adjacency matrix and stationary distribution. Verification of the Markov property for higher intersections. Construction of the isomorphism $\Phi : \Sigma_A \to \mathbb{T}^2$. Hölder continuity of Φ, exponential decay of correlations.

Construction of a Markov partition for the CAT map based on the stable and unstable manifolds of the fixed point.

April 16

Trapping region, attractor. Examples: sink. The solenoid map.

Transitive attractor. Discussion of the solenoid map: Markov partition, topological conjugacy with the double sided full shift.

Riemannian manifolds, tangent spaces, tangent maps. Diffeomorphisms, smooth dynamical systems. The spectrum of the tangent map for fixed points and periodic points. Hyperbolic fixed points. Two dimensional case: sinks, sources and saddles.

Statement of the unstable manifold theorem for 2d hyperbolic fixed points of C^1 diffeomorphisms, in particular saddles. Reduction to the case $|\lambda| > 2, |\mu| < \frac{1}{2}$. Stable and unstable cones in the tangent plane. Invariance and expansion properties of these cones w.r. to the tangent map at the fixed point. Choice of ε: these properties persist in a small neighborhood U.

April 22

Recap of the statement and the preparations for the proof of the unstable manifold theorem. Completing the proof: the complete metric space $\mathcal{H}$ of horizontal curves. The graph transform $\Phi : \mathcal{H} \to \mathcal{H}$ is a contraction. Banach fixed point theorem. The local unstable manifold as a fixed point of Φ.

Further discussion of the unstable manifold theorem: local and global unstable manifolds of a hyperbolic fixed point.

Definition of a hyperbolic set, invariant splitting of the tangent bundle. Formulation of the unstable manifold theorem and definition of local and global invariant manifolds for hyperbolic sets.

ω-limit sets, periodic, recurrent and non-wandering points. Anosov and Axiom A systems.

δ-chains and ε-shadowing. The shadowing property and its significance. Rotations do not have the shadowing property.

April 23

The doubling map has the shadowing property.

Proof of the shadowing property for the CAT map (and for hyperbolic systems).

Expansivity. Shift maps and hyperbolic systems are expansive.

Corollaries of the shadowing property and expansivity: uniqueness of the shadowing orbit. Periodic points are dense in the non-wandering set. Structural stability of hyperbolic sets.

Subadditive convergence theorem.

ε-covers, ε-nets and ε-separated sets in compact metric spaces. The dynamical metric d_n. Submultiplicativity of $C(n, \varepsilon, T)$.

April 29

Definition of the topological entropy. Determining the topological entropy for rotations and the full shift.

Measure theory recap: Lebesgue spaces. Finite partitions and finite sigma algebras: $\alpha \leq \beta$, $\alpha \vee \beta$, the metric $d(\alpha, \beta)$, $\alpha \perp \beta$.

Entropy of a finite partition as the expected information. Concavity of $\Phi(x) = -x \log x$ and some direct consequences.

Conditional entropy. Properties (i-iv) of $H(\alpha)$ and $H(\alpha|\beta)$.

Properties (v-xii) of $H(\alpha)$ and $H(\alpha|\beta)$.

April 30

The Rokhlin metric $\rho(\alpha, \beta)$. $\rho(\alpha, \beta)$ is absolutely continuous in $d(\alpha, \beta)$.

Definition of $h(T, \alpha)$. The limit exists by subadditivity.

Second proof on the existence of the limit defining $h(T, \alpha)$.

Definition of $h(T)$. For factors the entropy cannot increase, entropy is an isomorphism invariant.

Properties (1-9) of $h(T, \alpha)$ and $h(T)$.

Further properties: $|h(T, \alpha) - h(T, \beta)| \leq \rho(\alpha, \beta)$, $h(T, \alpha) = \lim_{n\to\infty} H(\alpha|T^{-1}\alpha_n)$ (where $\alpha_n = \bigvee_{i=0}^{n-1} T^{-i}\alpha$).

The notion of one-sided and two-sided generators, examples, the statement of the two versions of the Kolmogorov-Sinai theorem.