

Advanced theory of dynamical systems, Spring 2023

Homework problem set. Due on May 10, Wednesday

(Breakdown of points: 3 + 3 + 4 + 3 + 3 + 4 + 5 = 25)

1. Recall that, for an endomorphism (M, \mathcal{F}, T, μ) and a measurable $\hat{M} \subset M$ with $\mu(\hat{M}) > 0$, the *induced transformation* or *first return map* $\hat{T} : \hat{M} \rightarrow \hat{M}$ is defined as follows: for $x \in \hat{M}$ let

$$\hat{T}x = T^{r_{\hat{M}}(x)}x, \quad \text{where} \quad r_{\hat{M}}(x) = \min\{k \geq 1 | T^k x \in \hat{M}\}.$$

Let $\hat{\mu}$ be the conditional measure of μ on \hat{M} , that is, for measurable $B \subset \hat{M}$ let $\hat{\mu}(B) = \frac{\mu(B)}{\mu(\hat{M})}$. Show that $\hat{\mu}$ is indeed an invariant measure for the map \hat{T} . (*Comment: Do not assume that T is invertible! Consider the next problem, part (c) for a non-invertible example.*)

2. Figure out what is the first return map \hat{T} and what is the distribution of the first return time $r_{\hat{M}}$ (with respect to $\hat{\mu}$) in the following examples. For all of them $M = \mathbb{S}^1$, the circle, represented as the interval $[0, 1]$ with the two endpoints identified, \mathcal{F} is the sigma algebra of Lebesgue measurable sets and μ is the Lebesgue measure.

- (a) For some irrational $\alpha \in (0, \frac{1}{2})$, T is the rotation by α , ie. $Tx = x + \alpha \pmod{1}$, and $\hat{M} = [0, \alpha]$;
- (b) $Tx = x + \alpha \pmod{1}$, as above, but $\hat{M} = [0, \beta]$ for some $\beta \in (\alpha, 1 - \alpha)$;
- (c) $Tx = 2x \pmod{1}$ (the doubling map), and $\hat{M} = [\frac{1}{2}, 1]$.

3. (a) Let $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $T(x, y) = (x + \alpha, y)$, where $\alpha \in \mathbb{S}^1$ is irrational. Show that, given any $f \in C(\mathbb{T}^2)$, the Birkhoff averages converge uniformly, but the limit is typically not a constant function.
- (b) Consider $T : M \rightarrow M$ (with M compact metric space and T continuous) that is *topologically transitive*^a, and assume that for any $f \in C(M)$ the Birkhoff averages converge uniformly. Show that T is uniquely ergodic.
- (c) Let now $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $T(x, y) = (x + y \pmod{1}, y)$. Describe the set of invariant measures and the set of ergodic invariant measures (in class denoted as \mathcal{M}_{inv} and \mathcal{M}_{erg} , respectively) for this example.

4. Let $a_n \in \mathbb{R}$ be a bounded sequence. Show that $(ii) \Rightarrow (i)$ where

(i) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |a_k| = 0$;

(ii) $\exists J \subset \mathbb{Z}^+$ a sequence of zero density such that $\lim_{J \not\ni n \rightarrow \infty} a_n = 0$.

5. Show that, given any automorphism (M, \mathcal{F}, T, μ) , the following properties are equivalent: (i) T is weakly mixing, (ii) $T \times T$ is ergodic, (iii) $T \times T$ is weakly mixing. (Hint: to prove (ii) \rightarrow (i) consider sets of the form $A \times M$ and $A \times A$ in $M \times M$.)

6. Let (M, \mathcal{F}, T, μ) be an automorphism and consider the unitary operator $U_T : L^2_\mu \rightarrow L^2_\mu$, $U_T f = f \circ T$. Prove that

^aRecall that topological transitivity means that there exists some $x \in M$ such that the orbit of x is dense in M , that is, $\overline{\{T^n x | n \geq 0\}} = M$.

- (a) the spectrum of U_T is a subset of the complex unit circle \mathbb{S}^1 , and eigenfunctions corresponding to distinct eigenvalues of U_T are orthogonal.
- (b) 1 is always an eigenvalue of U_T , and T is ergodic iff 1 is a simple eigenvalue. *Below we restrict to the ergodic case.*
- (c) All eigenvalues of U_T are simple, and the modulus $|f|$ of any eigenfunction is μ -a.e. constant.
- (d) If λ, μ are eigenvalues, then so are $\bar{\lambda}$ (complex conjugate) and $\lambda \cdot \mu$.
7. Piecewise linear (strong) Markov maps $T : [0, 1] \rightarrow [0, 1]$ are associated to finite partitions $0 = a_0 < a_1 < a_2 < \dots < a_{q-1} < a_q = 1$ of the interval $[0, 1]$. When restricted to the subinterval $I_j = (a_{j-1}, a_j)$, the map is given by $T_j := T|_{I_j}$; $T_j x = \frac{x - a_{j-1}}{a_j - a_{j-1}}$, in particular, T_j is a *linear* one-to-one map of I_j onto $(0, 1)$ ($j = 1, 2, \dots, q$).
- (a) Show that the Lebesgue measure is invariant for T . (It is actually ergodic, which you do not have to prove, but may assume for the next question.)
- (b) For (Lebesgue) almost every $x \in (0, 1)$ the quantity $|(T^n)'(x)|$ – that is, the derivative of the n th iterate – is well-defined. Show that $|(T^n)'(x)|$ grows exponentially with the same rate λ for almost every $x \in (0, 1)$. Remark: in this simple context, λ is the occurrence of the „asymptotic expansion rate” or „Lyapunov-exponent”. Recall that a numerical sequence b_n grows exponentially with rate λ if $\lim_{n \rightarrow \infty} \frac{\ln b_n}{n} = \lambda$.
- (c) Show that the finite dimensional subspace of polynomials of degree at most K , to be denoted by E_K , is invariant under the action of the Perron-Frobenius operator, and determine the eigenvalues: $1, \beta_1, \dots, \beta_K$ (in decreasing order). Here β_1 is the rate of exponential decay on E_K . That is, for $f, g \in E_K$ (with arbitrary $K \geq 1$) we have $\text{Corr}_n(f, g) \leq C(\beta_1)^n$. The smaller $1 - \beta_1$, the slower is the decay.
- (d) Show that $\beta_1 \geq e^{-\lambda}$, where λ is the Lyapunov exponent.
- (e) Given any $M > 0$, construct a piecewise linear (strong) Markov map such that $\lambda > M$ but $1 - \beta_1 < M^{-1}$. That is, the Lyapunov exponent can be arbitrarily large and simultaneously, the exponential rate of correlation decay can be made arbitrarily slow. (*Hint:* to obtain a large M , the number of branches q has to be chosen large.)