Advanced theory of dynamical systems, Spring 2025

Homework problem set. Due on May 13, Tuesday

(*Breakdown of points:* $3+3+4+3+3+4+5=25$)

1. Recall that, for an endomorphism $(M,\mathcal{F},T,\mu)$ and a measurable $\hat{M}\subset M$ with $\mu(\hat{M})>0$, the *induced transformation* or *first return map* $\hat{T}:\hat{M}\to\hat{M}$ is defined as follows: for $x\in\hat{M}$ let

$$\hat{T}x = T^{r_{\hat{M}}(x)}x, \quad \text{where} \quad r_{\hat{M}}(x)=\min\{k\geq 1 | T^k x\in\hat{M}\}.$$

   Let $\hat{\mu}$ be the conditional measure of μ on $\hat{M}$, that is, for measurable $B\subset\hat{M}$ let $\hat{\mu}(B)=\frac{\mu(B)}{\mu(\hat{M})}$. Show that $\hat{\mu}$ is indeed an invariant measure for the map $\hat{T}$. (*Comment:* Do *not* assume that T is invertible! Consider the next problem, part (c) for a non-invertible example.)

2. Figure out what is the first return map $\hat{T}$ and what is the distribution of the first return time $r_{\hat{M}}$ (with respect to $\hat{\mu}$) in the following examples. For all of them $M=\mathbb{S}^1$, the circle, represented as the interval $[0,1]$ with the two endpoints identified, $\mathcal{F}$ is the sigma algebra of Lebesgue measurable sets and μ is the Lebesgue measure.
   (a) For some irrational $\alpha\in(0,\frac{1}{2})$, T is the rotation by α, ie. $Tx=x+\alpha \pmod 1$, and $\hat{M}=[0,\alpha)$;
   (b) $Tx=x+\alpha \pmod 1$, as above, but $\hat{M}=[0,\beta)$ for some $\beta\in(\alpha,1-\alpha)$;
   (c) $Tx=2x \pmod 1$ (the doubling map), and $\hat{M}=[\frac{1}{2},1)$.

3. (a) Let $T:\mathbb{T}^2\to\mathbb{T}^2$, $T(x,y)=(x+\alpha,y)$, where $\alpha\in\mathbb{S}^1$ is irrational. Show that, given any $f\in C(\mathbb{T}^2)$, the Birkhoff averages converge uniformly, but the limit is typically not a constant function.
   (b) Consider $T:M\to M$ (with M compact metric space and T continuous) that is *topologically transitive*[a], and assume that for any $f\in C(M)$ the Birkhoff averages converge uniformly. Show that T is uniquely ergodic.
   (c) Let now $T:\mathbb{T}^2\to\mathbb{T}^2$, $T(x,y)=(x+y \pmod 1, y)$. Describe the set of invariant measures and the set of ergodic invariant measures (in class denoted as $\mathcal{M}_{inv}$ and $\mathcal{M}_{erg}$, respectively) for this example.

4. Let $a_n\in\mathbb{R}$ be a bounded sequence. Show that $(ii)\Rightarrow(i)$ where
   (i) $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=0}^{n-1}|a_k|=0$;
   (ii) $\exists J\subset\mathbb{Z}^+$ a sequence of zero density such that $\lim\limits_{J\not\ni n\to\infty}a_n=0$.

5. Show that, given any automorphism $(M,\mathcal{F},T,\mu)$, the following properties are equivalent: (i) T is weakly mixing, (ii) $T\times T$ is ergodic, (iii) $T\times T$ is weakly mixing. (Hint: to prove (ii)→(i) consider sets of the form $A\times M$ and $A\times A$ in $M\times M$.)

6. Let $(M,\mathcal{F},T,\mu)$ be an automorphism and consider the unitary operator $U_T:L^2_\mu\to L^2_\mu$, $U_T f=f\circ T$. Prove that

[a] Recall that topological transitivity means that there exists some $x\in M$ such that the orbit of x is dense in M, that is, $\overline{\{T^n x\,|\,n\geq 0\}}=M$.

(a) the spectrum of U_T is a subset of the complex unit circle $\mathbb{S}^1$, and eigenfunctions corresponding to distinct eigenvalues of U_T are orthogonal.

(b) 1 is always an eigenvalue of U_T, and T is ergodic iff 1 is a simple eigenvalue. *Below we restrict to the ergodic case.*

(c) All eigenvalues of U_T are simple , and the modulus $|f|$ of any eigenfunction is μ-a.e. constant.

(d) If λ, μ are eigenvalues, then so are $\overline{\lambda}$ (complex conjugate) and $\lambda \cdot \mu$.

7. Piecewise linear (strong) Markov maps $T : [0,1] \to [0,1]$ are associated to finite partitions $0 = a_0 < a_1 < a_2 < \cdots < a_{q-1} < a_q = 1$ of the interval $[0,1]$. When restricted to the subinterval $I_j = (a_{j-1}, a_j)$, the map is given by $T_j := T|_{I_j}$; $T_j x = \frac{x - a_{j-1}}{a_j - a_{j-1}}$, in particular, T_j is a *linear* one-to-one map of I_j onto $(0,1)$ $(j = 1, 2, \ldots q)$.

   (a) Show that the Lebesgue measure is invariant for T. (It is actually ergodic, which you do not have to prove, but may assume for the next question.)

   (b) For (Lebesgue) almost every $x \in (0,1)$ the quantity $|(T^n)'(x)|$ – that is, the derivative of the nth iterate – is well-defined. Show that $|(T^n)'(x)|$ grows exponentially with the same rate λ for almost every $x \in (0,1)$. Remark: in this simple context, λ is the occurrence of the „asymptotic expansion rate" or „Lyapunov-exponent". Recall that a numerical sequence b_n grows exponentially with rate λ if $\lim_{n\to\infty} \frac{\ln b_n}{n} = \lambda$.

   (c) Show that the finite dimensional subspace of polynomials of degree at most K, to be denoted by E_K, is invariant under the action of the Perron-Frobenius operator, and determine the eigenvalues: $1, \beta_1, \ldots, \beta_K$ (in decreasing order). Here β_1 is the rate of exponential decay on E_K. That is, for $f, g \in E_K$ (with arbitrary $K \geq 1$) we have $Corr_n(f,g) \leq C(\beta_1)^n$. The smaller $1 - \beta_1$, the slower is the decay.

   (d) Show that $\beta_1 \geq e^{-\lambda}$, where λ is the Lyapunov exponent.

   (e) Given any $M > 0$, construct a piecewise linear (strong) Markov map such that $\lambda > M$ but $1 - \beta_1 < M^{-1}$. That is, the Lyapunov exponent can be arbitrarily large and simultaneously, the exponential rate of correlation decay can be made arbitrarily slow. (*Hint:* to obtain a large M, the number of branches q has to be chosen large.)