

Markov Chains and Dynamical Systems, Fall 2025

Homework problem set #4 . Due on November 24, Monday

1. Consider the one dimensional map $T_\lambda : \mathbb{R} \rightarrow \mathbb{R}$, $T_\lambda x = x^3 - \lambda x$, where the parameter λ satisfies $-\infty < \lambda \leq 1$. Investigate the λ -dependence of
 - (a) the fixed points and their stability properties.
 - (b) the asymptotic behavior of the orbit $T_\lambda^n x$, $n \geq 0$ for any initial condition $x \in \mathbb{R}$.
2.
 - (a) Prove that $\log_2(3)$ is an irrational number.
 - (b) For some $m \geq 2$, let $w = w_1 w_2 \dots w_m = 1 w_2 \dots w_m$ be a binary word of length m that starts with 1 but otherwise arbitrary, that is, $w_1 = 1$ while $w_i \in \{0, 1\}$, $i = 2, \dots, m$. Show that there exists some $n \geq 1$ such that the first m digits in the binary code of 3^n coincide with w .
 - (c) Consider the first two digits in the binary code for the sequence of numbers $3, 9, 27, \dots, 3^n, \dots$. There are two options: 10 and 11. Which one is more frequent? (Compute the asymptotic frequencies of these two options.)
3. Recall that, given two dynamical systems $T_1 : M_1 \rightarrow M_1$ and $T_2 : M_2 \rightarrow M_2$,
 - T_2 is a *factor* of T_1 if there is a map $\Phi : M_1 \rightarrow M_2$ such that $\Phi \circ T_1 = T_2 \circ \Phi$. In such a case, Φ is called a *semiconjugacy*.
 - If, moreover, Φ is a bijection (one-to-one and onto) then it is called a *conjugacy* (and the two dynamical systems are *conjugate*).

For any $\alpha \in (0, 1)$, let $R_\alpha : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ denote the rotation of the circle by α . Given α , let $\beta = 2\alpha \pmod{1}$ and $\gamma = 1 - \alpha$.

- (a) Show that R_β is a factor of R_α . (Moreover, the semiconjugacy is continuous).
 - (b) Show that R_γ and R_α are conjugate. (Moreover, the conjugacy is a homeomorphism of $\mathbb{S}^1$, that is, a continuous bijection with a continuous inverse).
4. Consider the doubling map $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, $Tx = 2x \pmod{1}$. Let $D \subset \mathbb{S}^1$ denote the set of points x such that $\{T^n x | n = 0, 1, 2, \dots\}$ is dense in $\mathbb{S}^1$. Prove that D is uncountable.

Bonus problem: Prove that $\lambda(D) = 1$ (where λ is the Lebesgue measure).

5. Consider the logistic map $T_a x = ax(1 - x)$ with $a > 2 + \sqrt{5}$. Show that there exists some $\lambda > 1$ such that $|T'_a(x)| > \lambda$ whenever $x \in I_0 \cup I_1$. (We use the notation $T_a^{-1}[0, 1] = I_0 \cup I_1$, where I_0 and I_1 are disjoint intervals.)