① $c=3,\ L=2,\ f(x)\equiv 0$ (3p)

$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$

$g(x)=\sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{3\pi}{2}x\right)+\sin(2\pi x) = \frac{1}{2}\left[\sin(2\pi x)+\sin(-\pi x)\right]+\sin 2\pi x =$ (4p)

$= \frac{3}{2}\sin(2\pi x) - \frac{1}{2}\sin(\pi x) = \sum_{k=1}^{\infty} D_k \sin\left(\frac{k\pi}{2}x\right) \Rightarrow D_2 = -\frac{1}{2},\ D_4=\frac{3}{2}$ (4p)

egyébként $D_k = 0$

$B_k = \dfrac{D_k}{\frac{k\pi c}{L}} = \dfrac{2\cdot D_k}{3k\pi}$; $B_2 = \dfrac{-1}{6\pi}$; $B_4 = \dfrac{1}{4\pi}$ egyébként $B_k = 0$ (2p)

$u(x,t) = \sum_{k=1}^{\infty}\sin\left(\frac{k\pi}{2}x\right)\cdot B_k \sin\left(\frac{3k\pi}{2}t\right) = -\frac{1}{6\pi}\sin(\pi x)\sin(3\pi t)+\frac{1}{4\pi}\sin(2\pi x)\sin(6\pi t)$ (2p)

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② $\begin{pmatrix}1&1&2\\0&1&1\\-1&3&2\\2&-2&0\end{pmatrix} \sim \begin{pmatrix}1&1&2\\0&1&1\\0&4&4\\0&-4&-4\end{pmatrix} \sim \begin{pmatrix}1&1&2\\0&1&1\\0&0&0\\0&0&0\end{pmatrix}$

3s. + 1.s.
4s. + (−2).1s

3s. − 4.(2s)
4s. + 4.(2s.)

(5p)

(a) 1. és 2. oszlop pivot $\Rightarrow$

$\Rightarrow \mathcal{B} = \{\underline{v}_1, \underline{v}_2\}$ (4p)

(b) kételemű bázis $\Rightarrow \dim V = 2$ (3p); $\dim V^{\perp} = 4 - \dim V = 2$ (3p)

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③ $0 = \begin{vmatrix}52-\lambda & -36\\ -36 & 73-\lambda\end{vmatrix} = (73-\lambda)(52-\lambda)-(-36)^2 = 3796 - 125\lambda + \lambda^2 - 1296 =$

$= \lambda^2 - 125\lambda + 2500 = (\lambda-25)(\lambda-100)$ $\quad \lambda_1 = 25,\ \lambda_2 = 100$

poz. def, mert két poz. sajátérték (4p)

$\underline{\lambda_1 = 25}$:

$\begin{pmatrix}27&-36\\-36&48\end{pmatrix} \sim \begin{pmatrix}3&-4\\-3&4\end{pmatrix} \sim \begin{pmatrix}3&-4\\0&0\end{pmatrix} \Rightarrow \underline{v}_1 = \begin{pmatrix}t\\ \frac{3}{4}t\end{pmatrix}$; $\underline{v}_1^0 = \begin{pmatrix}4/5\\3/5\end{pmatrix}$ (3p)

$\underline{\lambda_2 = 100} \Rightarrow \underline{v}_2^0 \perp \underline{v}_1^0$, így: $\underline{v}_2^0 = \begin{pmatrix}-3/5\\4/5\end{pmatrix}$ (2p)

$Q = \begin{pmatrix}4/5&-3/5\\3/5&4/5\end{pmatrix}$ $\quad Q^T = \begin{pmatrix}4/5&3/5\\-3/5&4/5\end{pmatrix}$ $\quad D = \begin{pmatrix}25&0\\0&100\end{pmatrix}$, $\sqrt{D} = \begin{pmatrix}5&0\\0&10\end{pmatrix}$ (2p)

$B = Q\sqrt{D}Q^T = \begin{pmatrix}4/5&-3/5\\3/5&4/5\end{pmatrix}\begin{pmatrix}5&0\\0&10\end{pmatrix}\begin{pmatrix}4/5&3/5\\-3/5&4/5\end{pmatrix} = \begin{pmatrix}34/5&-12/5\\-12/5&41/5\end{pmatrix}$ (4p)

$\begin{pmatrix}4/5&3/5\\-3/5&4/5\end{pmatrix}$

$\begin{pmatrix}5&0\\0&10\end{pmatrix}\begin{pmatrix}4&3\\-6&8\end{pmatrix}$

$\begin{pmatrix}4&3\\-6&8\end{pmatrix}$

$\begin{pmatrix}4/5&-3/5\\3/5&4/5\end{pmatrix}\begin{pmatrix}34/5&-12/5\\-12/5&41/5\end{pmatrix}$

(4.)

$$\operatorname{curl} F = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz+e^{xy}y & x^2z+e^{xy}x & x^2y+2z \end{vmatrix} = \underline{i}\left(x^2-x^2\right) - \underline{j}\left(\cancel{}\, 2xy-2xy\right) + \underline{k}\Big(2xz+e^{xy}+$$
$$+xye^{xy}-\left(2xz+e^{xy}+xye^{xy}\right)\Big) = \boxed{\underline{0}}$$

és mivel $\underline{F}(\underline{r})$ mindenütt sima, $\underline{F}(\underline{r})$ konzervatív. 5p

(a) $U(x,y,z)=?$ $\quad \dfrac{\partial U}{\partial x} = 2xyz + e^{xy}y \Rightarrow U(x,y,z) = x^2yz + e^{xy} + C_1(y,z)$

$x^2z + e^{xy}x = \dfrac{\partial U}{\partial y} = x^2z + xe^{xy} + \dfrac{\partial C_1}{\partial y} \Rightarrow \dfrac{\partial C_1}{\partial y} = 0$, így $C_1(y,z) = C_1(z)$

$x^2y + 2z = \dfrac{\partial U}{\partial z} = x^2y + \dfrac{dC_1}{dz} \Rightarrow \dfrac{dC_1}{dz} = 2z \Rightarrow C_1 = z^2 + C$ 6p

tehát: $U(x,y,z) = x^2yz + e^{xy} + z^2 + C$

(b) mivel $\underline{F}$ konzervatív és γ zárt, $\displaystyle\int_\gamma \underline{F}\, d\underline{r} = 0$ 4p

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(5.) $\underline{G}(\underline{r}(u,v)) = (u-v,\ u+v,\ 3u-4v)$ 3p

$\underline{r}_u = (1,0,3)$; ~~$\underline{r}_v = (0,0,-4$~~ $\underline{r}_v = (0,1,-4)$ $\qquad \underline{r}_u \times \underline{r}_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 3 \\ 0 & 1 & -4 \end{vmatrix} = -3\underline{i} + 4\underline{j} + \underline{k}$ 4p

lefelé mutató normális $\Rightarrow \underline{n} = -(\underline{r}_u \times \underline{r}_v) = (3,-4,-1)$ 2p

$$\iint_S \underline{G}\, d\underline{A} = \int_0^1\!\!\int_0^1 \underline{G}(\underline{r}(u,v)) \cdot \underline{n}(u,v)\, du\, dv = \int_0^1\!\!\int_0^1 (3u-3v-4u-4v-3u+4v)\, du\, dv$$ 3p

$$= \int_0^1\!\!\int_0^1 (-4u-3v)\, du\, dv = -\int_0^1 4u\, du - 3\int_0^1 v\, dv = -4\left[\frac{u^2}{2}\right]_0^1 - 3\left[\frac{v^2}{2}\right]_0^1 = -2-\frac{3}{2} = -\frac{7}{2}$$ 3p

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(6.)

$$\operatorname{curl} F = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ x^2+z^2 & xyz & x+z \end{vmatrix} = \underline{i}(0-xy) - \underline{j}(1-2z) + \underline{k}(yz-0) = (-xy,\ 2z-1,\ yz)$$ 6p

$\operatorname{curl} F|_P = (0,3,0)$ 4p $\Rightarrow \underline{n} = \dfrac{\operatorname{curl} F|_P}{|\operatorname{curl} F|_P|} = (0,1,0)$ 5p