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Complementarity in Quantum Systems

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Abstract: Reduction of a state of a quantum system to a subsystem gives partial quantum information about the true state of the total system. Two subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of $B(\mathcal{H})$ are called complementary if the traceless subspaces of $\mathcal{A}_1$ and $\mathcal{A}_2$ are orthogonal (with respect to the Hilbert-Schmidt inner product). When both subalgebras are maximal Abelian, then the concept reduces to complementary observables or mutually unbiased bases. In the paper several characterizations of complementary subalgebras are given in the general case and several examples are presented. For a 4-level quantum system, the structure of complementary subalgebras can be described very well, the Cartan decomposition of unitaries plays a role. It turns out that a measurement corresponding to the Bell basis is complementary to any local measurement of the two-qubit-system.

Key words: Entropic uncertainty relation, mutually unbiased basis, CAR algebra, commuting squares, complementarity, Cartan decomposition, Bell states.

The study of complementary observables goes back to early quantum mechanics. Position and momentum are the typical examples of complementary observables and the main subject was the joint measurement and the uncertainty [8, 9]. In the setting of finite dimensional Hilbert space and in a mathematically rigorous approach, the paper [24] of Schwinger might have been the first in 1960. The goal of that paper is the finite dimensional approximation of the canonical commutation relation. An observable of a finite system can be identified with a basis of the Hilbert space through the spectral theorem [1] and instead of complementarity the expression "mutually unbiased" became popular [29]. The maximum number of mutually unbiased bases is still and open question [22], nevertheless such bases are used in several contexts, state determination, the "Mean King's problem", quantum cryptography etc. [12, 13, 6].

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Motivated by the frequent use of mutually unbiased bases and complementary reductions of two qubits [19, 21], the goal of this paper is a general study of complementary subalgebras. The particular case, when the subalgebras are maximal Abelian, corresponds to complementary observables, or mutually unbiased bases. This case has been studied in the literature by many people. If the reduction of a quantum state to a subalgebra is known to us, then this means a partial information about the state. The concept of complementarity of two subsystems means heuristically that the partial information provided jointly by the two subsystems is the largest when it is compared with the information content of the two subsystems [29].

The paper is organized in the following way. First the entropic uncertainty relation of Maasen and Uffink is reviewed as a motivation for the concept of complementarity (of observables or basis). Then the complementarity of observables is reformulated in terms of commutative subalgebras. This reformulation leads to the complementarity of more general subalgebras (corresponding to a subsystem of a quantum system). It turns out that complementarity is a common generalization of the ordinary tensor product and the twisted fermionic tensor product. When two subalgebras are unitarily equivalent, complementarity can be read out from the unitary when it is viewed as a block-matrix. A modification of the construction of complementary bases (going back to Schwinger) yields examples of complementary subalgebras in arbitrary dimension. The maximal number of complementary subalgebras remains an open question, however, the case of 4-level quantum system is analyzed in details. It turns out that a measurement corresponding to the Bell basis is complementary to any local measurement of the two-qubit-system.

1 Complementary observables

Let A and B be two self-adjoint operators on a finite dimensional Hilbert space. If $A=\sum_i \lambda_i^A P_i^A$ and $B=\sum_i \lambda_i^B P_i^B$ are their spectral decompositions, then

$$H(A,\varphi)=\sum_i \eta(\varphi(P_i^A)) \quad \text{and} \quad H(B,\varphi)=\sum_i \eta(\varphi(P_i^B))$$

are the **entropies** of A and B in a state φ. ($\eta(t)$ is the function $-t\log t$.)

Assume that the eigenvalues of A and B are free from multiplicities. If these observables share a common eigenvector and the system is prepared in the corresponding state, then the measurement of both A and B leads to a sharp distribution and one cannot speak of uncertainty. In order to exclude this case, let (e_i) be an orthonormal basis consisting of eigenvectors of A, let (f_i) be a similar basis for B and we suppose that

$$c^2 := \sup\{|\langle e_i, f_j\rangle|^2 : i,j\} \tag{1}$$

is strictly smaller than 1. Then $H(A,\varphi)+H(B,\varphi)>0$ for every pure state φ. Since the left-hand-side is concave in φ, it follows that $H(A,\varphi)+H(B,\varphi)>0$ for any state

φ. This inequality is a sort of uncertainty relation. The lower bound was conjectured in [14] and proven by Maasen and Uffink in [16].

Theorem 1 *With the notation above the uncertainty relation*

$$H(A,\varphi) + H(B,\varphi) \geq -2\log c$$

holds.

Let n be the dimension of the underlying Hilbert space. We may assume that φ is a pure state corresponding to a vector Φ. Then $\varphi(P_i^A) = |\langle e_i, \Phi\rangle|^2$ and $\varphi(P_i^B) = |\langle f_i, \Phi\rangle|^2$.

The $n \times n$ matrix $T_{i,j} := (\langle f_i, e_j\rangle)_{i,j}$ is unitary and T sends the vector

$$f := (\langle e_1, \Phi\rangle, \langle e_2, \Phi\rangle, \ldots, \langle e_n, \Phi\rangle)$$

into

$$Tf = (\langle f_1, \Phi\rangle, \langle f_2, \Phi\rangle, \ldots, \langle f_n, \Phi\rangle)\,.$$

The vectors f and Tf are elements of $\mathbb{C}^n$ and this space may be endowed with different L^p norms. Using interpolation theory we shall estimate the norm of the linear transformation T with respect to different L^p norms. Since T is a unitary

$$\|g\|_2 = \|Tg\|_2 \qquad (g \in \mathbb{C}^n)\,.$$

With the notation (1) we have also

$$\|Tg\|_\infty \leq c\|g\|_1 \qquad (g \in \mathbb{C}^n)\,.$$

Let us set

$$N(p,p') = \sup\{\|Tg\|_p/\|g\|_{p'} \colon g \in \mathbb{C}^n, \quad g \neq 0\}$$

for $1 \leq p \leq \infty$ and $1 \leq p' \leq \infty$. The **Riesz–Thorin convexity theorem** says that the function

$$(t,s) \mapsto \log N(t^{-1}, s^{-1}) \tag{2}$$

is convex on $[0,1] \times [0,1]$ (where 0^{-1} is understood to be ∞). Application of convexity of (2) on the segment $[(0,1),(1/2,1/2)]$ yields

$$\|Tg\|_{2/\lambda} \leq c^{1-\lambda}\|g\|_\mu \qquad (g \in \mathbb{C}^n)\,,$$

where $0 < \lambda < 1$ and $\mu = (1-\lambda/2)^{-1}$. This is rewritten by means of a more convenient parameterization in the form

$$\|Tg\|_p \leq c^{1-2/p}\|g\|_q \qquad (g \in \mathbb{C}^n)\,,$$

where $2 < p < \infty$ and $p^{-1} + q^{-1} = 1$. Consequently

$$\log\|Tf\|_p \leq \left(1 - \frac{2}{p}\right)\log c + \log\|f\|_q\,. \tag{3}$$

One checks easily that

$$\left.\frac{d\log\|Tf\|_p}{dp}\right|_{p=2} = -\frac{1}{4}H(B,\varphi) \quad \text{and} \quad \left.\frac{d\log\|f\|_q}{dp}\right|_{p=2} = \frac{1}{4}H(A,\varphi)\,.$$

Hence dividing (3) by $p-2$ and letting $p \searrow 2$ we obtain

$$-\tfrac{1}{4}H(B,\varphi) \le \tfrac{1}{2}\log c + \tfrac{1}{4}H(A,\varphi)$$

which proves the theorem for a pure state.

Concavity of the left hand side of the stated inequality in φ ensures the lower estimate for mixed states. □

The theorem can be formulated in an algebraic language. Let $\mathcal{A}$ and $\mathcal{B}$ be maximal Abelian subalgebras of the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices. Set

$$c^2 := \sup\{\mathrm{Tr}\,PQ : P \in \mathcal{A}, Q \in \mathcal{B} \text{ are minimal projections}\}. \tag{4}$$

The theorem tells that

$$H(\varphi|\mathcal{A}) + H(\varphi|\mathcal{B}) \ge -2\log c\,. \tag{5}$$

Both the definition of c and the statement are formulated without the underlying Hilbert space.

Question 1 Can we make the proof of (5) without using the Hilbert space?

Let A and B self-adjoint operators with eigenvectors (e_j) and (f_i), respectively and let φ be the pure state corresponding to e_1. Then $H(A,\varphi) = 0$ and $H(B,\varphi) = \log n$. Hence this example shows that the lower bound for the entropy sum in Theorem 1 is sharp. If (6) holds then the pair (A, B) of observables are called **complementary** [1]. According to another terminology, the bases $(e_j)_j$ and $(f_k)_k$ are called **mutually unbiased** if (6) holds. Mutually unbiased bases appeared in a different setting in the paper [12, 29], where state determination was discussed.

The lower bound in the uncertainty (5) is the largest if c^2 is the smallest. Since $n^2c^2 \ge n$, the smallest value of c^2 is $1/n$. This happens if and only if

$$|\langle e_j, f_k\rangle|^2 = n^{-1} \qquad (j,k = 1,2,\dots,n)\,, \tag{6}$$

that is, the two bases are mutually unbiased. This is an extremal property of the mutually unbiased bases. The largest lower bound is attained if ϕ is a vector state generated by one of the basis vectors.

The complementarity of observables is also the property of the spectral measures associated with them. Therefore the extension to POVM's is natural. For a POVM $\mathcal{E} \equiv (E_i)_i$ and for a unit vector Φ, we define an entropy quantity as

$$H(\mathcal{E},\Phi) = \sum_i \eta(\langle\Phi, E_i\Phi\rangle).$$