9. gyakorló feladatsor néhány feladatának a megoldása

① $f, g \in L^1 \Rightarrow |f^2+g^2|^{1/2} \in L^1$

$a, b \in \mathbb{R}^+$: $a^2+b^2 \leq (a+b)^2 \Rightarrow |f^2+g^2|^{1/2} \leq |f|+|g| \in L^1$

$\Rightarrow |f^2+g^2| \in L^1$

② $a>0 \Rightarrow$ (a) $\ln\frac{1}{x} \in L^p(0,a)$ $\quad p \in (0,\infty)$

(b) $e^{1/x} \notin L^p(0,a)$ $\quad p \in (0,\infty)$

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a) $\ln\frac{1}{x}$ $\quad x=0$ körüliben problémás, de belátható, hogy elég kicsi x-ekre $\left(\ln\frac{1}{x}\right)^p \leq \frac{1}{\sqrt{x}} \Rightarrow \int_0^a \left(\ln\frac{1}{x}\right)^p dx \leq \int_0^a \frac{1}{\sqrt{x}}dx < \infty$

b) $\forall \alpha>0$ esetén $t^\alpha e^{1/t} \to \infty$, ha $t \to 0$

↳ pl: $\alpha = \frac{1}{p}$-re, elég kicsi x-re $\quad x^{1/p}e^{1/x} > 1$

$$\Updownarrow$$

$$e^{1/x} > \frac{1}{\sqrt[p]{x}}$$

$$\Updownarrow$$

$$\left(e^{1/x}\right)^p > \frac{1}{x}$$

de $\frac{1}{x} \notin L^p(0,a) \Rightarrow e^{1/x} \notin L^p(0,a)$

(7) Bizonyítsuk be, hogy az L^1-beli konvergenciából következik a mértékbeli konvergencia. Mutassunk ellenpéldát arra, hogy a megfordítás nem igaz.

Indirekt tfh $f_n \xrightarrow{L_1} f$, de $f_n \overset{\mu}{\not\to} f$

$\hookrightarrow$ $\exists\, \varepsilon>0$, hogy $\forall\, N \geq 1$-re $\exists\, n>N$, hogy

$$\mu\{x : |f_n(x)-f(x)|>\varepsilon\}>\varepsilon$$

Ekkor

$$\int_X |f_n - f|\, d\mu \geq \varepsilon\, \mu\{x : |f_n(x)-f(x)|>\varepsilon\} \geq \varepsilon^2 > 0$$

$$\text{vagyis } f_n \overset{L_1}{\not\to} f \quad \text{: } \lightning$$

Ellenpélda arra, hogy a megfordítás nem igaz:

$X := [0,1]$, μ: Lebesgue-mérték

$$f_n := n\chi_{(0,1/n)} \implies f_n \to f \equiv 0 \text{ mértékben}$$

$$\text{de } \int_{[0,1]} |f_n - f|\, d\mu = 1 \quad \forall n \in \mathbb{N}$$

$$\hookrightarrow f_n \overset{L_1}{\not\to} f. \quad \circ$$

(8) $f \in L^2(\mathbb{R}) \implies \lim_{n\to\infty} \int_n^{n+1} f = 0$.

$$\left|\int_n^{n+1} f\right| \leq \int_n^{n+1} |f| = \int_n^{n+1} |f|\cdot 1 \underset{\substack{\uparrow \\ \text{Schwarz-egy.}}}{\leq} \left(\int_n^{n+1} |f|^2\right)^{1/2} \left(\int_n^{n+1} 1^2\right)^{1/2} = \left(\int_n^{n+1} |f|^2\right)^{1/2}$$

$\mu(E) := \int_E |f|^2$ mérték $\mathbb{R}$-en, mely véges: $\mu(\mathbb{R}) = \int_{\mathbb{R}} |f|^2 < \infty$.

$$\hookrightarrow \int_n^{n+1} |f|^2 = \mu([n,n+1]) \leq \mu([n,\infty)) \xrightarrow[n\to\infty]{} \mu\left(\bigcap_{n=1}^{\infty} [n,\infty)\right) = \mu(\emptyset) = 0 \quad \circ!$$

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(9) $f \geq 0$ Lebesgue-mérhető $[0,1]$-en és

$$\int_{[0,1]} f^2 = \int_{[0,1]} f^3 = \int_{[0,1]} f^4 < \infty. \quad \text{akkor} \quad f = f^2 \text{ m.m.}$$

$$\int_{[0,1]} f \cdot f^2 \underset{\text{Schwarz}}{\leq} \left[\int_{[0,1]} f^2 \right]^{1/2} \left[\int_{[0,1]} f^4 \right]^{1/2} = \int_{[0,1]} f^2$$

Tudjuk, hogy pontosan akkor lesz egyenlőség a Schwarz-egyenlőtlenségben, ha $a f = b f^2$ m.m valamely $a, b \in \mathbb{R}$-re

$$\underset{\text{feltétel}}{\hookrightarrow} \quad a = b = 1 \quad : \quad f = f^2 \text{ m.m.}$$

Vagyis $f = f^2$ mm $\Rightarrow$ f 0 v 1 mm $\Rightarrow$ $f = \chi_A$ valamely $A \subset [0,1]$.

(10) $p > 1$, $f \in L^p([-1,1]) \Rightarrow$ a) $f \in L^1([-1,1])$

b) $I_n = (-1/n, 1/n)$, $\gamma = \frac{p-1}{p}$

$$\Rightarrow \lim_{n \to \infty} n^\gamma \int_{I_n} |f| = 0$$

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a) $$\int_{[-1,1]} |f| \cdot 1 \underset{\text{Hölder}}{\leq} \left[\int_{[-1,1]} |f|^p \right]^{1/p} \left[\int_{[-1,1]} 1^q \right]^{1/q} = 2^{1/q} \cdot \|f\|_p < \infty$$

ahol $\frac{1}{p} + \frac{1}{q} = 1$, azaz $q = \frac{p}{p-1}$.

b) $q = \frac{p}{p-1} \Rightarrow \gamma = \frac{1}{q}$

$$\int_{I_n} |f| \cdot 1 \underset{\text{Hölder}}{\leq} \left(\int_{I_n} |f|^p \right)^{1/p} \left(\int_{I_n} 1^q \right)^{1/q} = \left(\frac{2}{n} \right)^\gamma \left[\int_{I_n} |f|^p \right]^{1/p}$$

$$\hookrightarrow n^\gamma \int_{I_n} |f| \leq 2^\gamma \left[\int_{I_n} |f|^p \right]^{1/p}$$

(10) folyt

$$\mu(E) := \int_E |f|^p \quad \text{mérték } [-1,1]\text{-en}$$

$$\mu([-1,1]) = \int_{[-1,1]} |f|^p < \infty \implies \mu \text{ véges mérték}$$

$$\int_{I_n} |f|^p = \mu(I_n) \xrightarrow[n\to\infty]{} \mu\left(\bigcap_{n=1}^{\infty} I_n\right) = \mu(\{0\}) = \int_{\{0\}} |f|^p = 0$$

(11) $f_n \in L^2(a,b)$, $n \in \mathbb{N}$, $f \in L^2(a,b)$: $\lim_n \|f_n - f\| = 0$

$\implies$ a) $\int_a^b f^2 = \lim_n \int_a^b f_n^2$

b) $\int_a^t f = \lim_n \int_a^t f_n$, $a \le t \le b$

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a) $$\big| \|f\|_2 - \|f_n\|_2 \big| \le \|f - f_n\|_2 \to 0 \implies \int_a^b f^2 = \lim_n \int f_n^2$$

b) $$\left| \int_a^t f - \int_a^t f_n \right| = \left| \int_a^b \chi_{(a,t)} (f - f_n) \right| \underset{\text{Hölder}}{\le} \sqrt{\int_a^b \chi_{(a,t)}^2} \sqrt{\int_a^b |f - f_n|^2} =$$

$$= \sqrt{t-a} \cdot \|f - f_n\|_2 \to 0$$

(12)
a) $\int_0^\pi x^{-1/4} \sin x \, dx \le \pi^{3/4}$

$$\int_0^\pi x^{-1/4} \sin x \, dx \underset{x\in(0,\pi)}{=} \int_0^\pi |x^{-1/4}| \cdot |\sin x| \, dx \underset{\text{Schwarz}}{\le} \left[\int_0^\pi x^{-1/2} dx \right]^{1/2} \cdot \left[\int_0^\pi \sin^2 x \, dx \right]^{1/2}$$

$$= \left\{ \left[2\sqrt{x} \right]_0^\pi \cdot \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right]_0^\pi \right\}^{1/2} = \left\{ 2\sqrt{\pi} \cdot \frac{1}{2}(\pi) \right\}^{1/2} = \pi^{3/4}$$

3/ (12) b) $\displaystyle\int_1^\infty \frac{\sqrt[3]{1+x}}{x^2}\,dx \le \sqrt[3]{6}$

$$\int_1^\infty \frac{\sqrt[3]{1+x}}{x}\cdot\frac{1}{x}\,dx \underset{\uparrow}{\le} \left[\int_1^\infty \frac{1+x}{x^3}\,dx\right]^{1/3}\left[\int_1^\infty \frac{1}{x^{3/2}}\,dx\right]^{2/3} =$$

Hölder $p=3 \rightsquigarrow \frac{1}{p}+\frac{1}{q}=1 \Rightarrow q=\frac{3}{2}$

$$= \left(\frac{1}{2}+1\right)^{1/3}\cdot 2^{2/3} = \sqrt[3]{6}$$

c) $\displaystyle\left(\int_0^1 \frac{x^3}{(1-x)^{1/5}}\,dx\right)^5 \le \frac{16}{81}$

$$\int_0^1 \frac{x^3}{(1-x)^{1/5}}\,dx \underset{\uparrow}{\le} \left(\int_0^1 (x^3)^5\,dx\right)^{1/5}\left(\int_0^1 \frac{1}{[(1-x)^{1/5}]^{5/4}}\,dx\right)^{4/5} =$$

Hölder $p=5$

$q=5/4$

$$= \left(\int_0^1 x^{15}\,dx\right)^{1/5}\left(\int_0^1 \frac{1}{(1-x)^{1/4}}\,dx\right)^{4/5} = \left(\frac{1}{16}\right)^{1/5}\cdot\left(\frac{4}{3}\right)^{4/5} = \left(\frac{16}{81}\right)^{1/5}.$$

d) $\displaystyle\int_0^1 \sqrt{x^4+4x^2+3}\,dx \le \frac{2\sqrt{10}}{3}$

$$\int_0^1 \sqrt{x^4+4x^2+3}\,dx = \int_0^1 \sqrt{x^2+3}\,\sqrt{x^2+1}\,dx \underset{\uparrow}{\le} \left(\int_0^1 (x^2+3)\,dx\right)^{1/2}\left(\int_0^1 (x^2+1)\,dx\right)^{1/2} =$$

Schwarz

$$= \sqrt{\frac{10}{3}}\sqrt{\frac{4}{3}} = \frac{2}{3}\sqrt{10}.$$

e) $$\int_0^{\pi/2} \sqrt{x \sin x}\, dx < \frac{\pi}{2\sqrt{2}}$$

$$\int_0^{\pi/2} \sqrt{x\sin x}\,dx \underset{\text{Schwarz}}{\le} \left(\int_0^{\pi/2} x\,dx\right)^{1/2}\left(\int_0^{\pi/2} \sin x\,dx\right)^{1/2} = \left(\frac{1}{2}\left(\frac{\pi}{2}\right)^2\right)^{1/2}\cdot 1^{1/2} = \frac{\pi}{2\sqrt{2}}.$$

(13) $f \in L^2(0,\pi) \Rightarrow \int_0^\pi (f(x)-\sin x)^2\,dx \le \frac{4}{9}$ és $\int_0^\pi (f(x)-\cos x)^2\,dx \le \frac{1}{9}$

inkonzisztensek.

$$\|\sin x - \cos x\|_2 = \|(f-\sin x)-(f-\cos x)\|_2 \le \|f-\sin x\|_2 + \|f-\cos x\|_2$$

$$\le \frac{2}{3} + \frac{1}{3} = 1$$

de $$\|\sin x - \cos x\|_2 = \left[\int_0^\pi \underbrace{(\sin x-\cos x)^2}_{1-2\sin x\cos x = 1-\sin 2x}\,dx\right]^{1/2} = \Bigg(\underbrace{\left[x+\frac{\cos 2x}{2}\right]_0^\pi}_{\pi}\Bigg)^{1/2} = \sqrt{\pi}$$

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