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Functional Analysis - Exercises 5

Solutions

(1) l_p is reflexive for every $p \in (1,\infty)$

$(Jx)(\varphi) := \varphi(x) \quad x \in X, \varphi \in X^*$ canonical embedding ($J: X \to X^{**}$)

recall: X is reflexive iff J is surjective

$\forall \varphi \in l_p^* \ \exists!\ y_\varphi \in l_q$ s.t. $\|y_\varphi\|_q = \|\varphi\|$ and

$\frac{1}{p}+\frac{1}{q}=1$

$$\sum_{n=1}^{\infty} (y_\varphi)_n x_n = \varphi(x) = (Jx)(\varphi)$$

$x = (x_1, x_2, \ldots) \in l_q$

$y_\varphi = ((y_\varphi)_1, (y_\varphi)_2, \ldots) \in l_q$

Now let $f \in l_p^{**}$. Since $l_p^* \cong l_q \Rightarrow l_p^{**} \cong l_q^*$, so

f can be considered as a bounded linear functional on l_q.

$\Rightarrow \exists \ z_f = ((z_f)_1, (z_f)_2, \ldots) \in l_p$ s.t.

$$\underset{l_q}{f(y_\varphi)} = \sum_{n=1}^{\infty} (z_f)_n \cdot (y_\varphi)_n = \varphi(z_f) = (Jz_f)(y_\varphi)$$

Since every $y \in l_q$ is equal to y_φ for some $\varphi \in l_p^*$, this shows that $Jz_f = f$

$\Downarrow$

Every $f \in l_p^{**}$ can be obtained in the form $f = Jx$ for some $x \in l_p$

$\Downarrow$

J is surjective $\quad$!

2)

② Let X be a normed space.

a) If X is finite dimensional then it is reflexive.

Let $\mathcal{J}: X \to X^{**}$ is the canonical embedding $\Rightarrow$ $\mathcal{J}$ is injective

If X is finite dimensional $\Longrightarrow$ $\dim X = \dim X^* = \dim X^{**}$

$\hookrightarrow$ an injective map from X to X^{**} is also surjective

$\Downarrow$

$\mathcal{J}$ is surjective $\Rightarrow$ X is reflexive.

b) If X is reflexive and separable then X^* is separable.

If X is reflexive, then X^{**} is isometrically isomorphic to X under the canonical embedding $\mathcal{J}$.

$\Rightarrow$ if D is a countable dense set in X $\Rightarrow$ $\mathcal{J}(D)$ is a countable dense set in X^{**}

$\Downarrow$

X^{**} is separable

We proved: X^* is separable $\Rightarrow$ X is separable

$\hookrightarrow$ $X^{**} = (X^*)^*$ is separable $\Rightarrow$ X^* is separable

$\circ$!

3/ c) If X is reflexive then X^* is reflexive.

Let $J_X : X \to X^{**}$ and $J_{X^*} : X^* \to (X^*)^{**} = X^{***}$

be canonical embeddings.

We have to show that J_{X^*} is surjective if ~~J_X~~ X is reflexive, i.e.

$$\forall F \in X^{***} \quad \exists \; f_F \in X^* \text{ s.t. } J_{X^*} f_F = F .$$

X is reflexive $\Rightarrow \forall \varphi \in X^{**} \; \exists! \; x_\varphi \in X$ s.t. $\varphi = J_X x_\varphi$

For $F \in X^{***}$ and $\varphi \in X^{**}$:

$$F(\varphi) = F(J_X x_\varphi) = (F \circ J_X)(x_\varphi) \qquad (*)$$

Note: $F \circ J_X \in X^*$, and we have, ~~for~~ for any $\varphi \in X^{**}$

$$J_{X^*}(F \circ J_X)(\varphi) = \varphi(F \circ J_X) = (J_X x_\varphi)(F \circ J_X) = (F \circ J_X)(x_\varphi) \underset{(*)}{=} F(\varphi)$$

$$\Rightarrow F = J_{X^*}(\underbrace{F \circ J_X}_{\equiv f_F})$$

d) If X is Banach and X^* is reflexive $\Rightarrow X$ is reflexive

Assume, on the contrary, that X^* is reflexive but X is not.

$\Rightarrow \exists \; \varphi \in X^{**} \setminus J_X(X)$. Since X is a Banach space, and J_X is an isometry ~~is~~, $J_X(X)$ is also a Banach space $\Rightarrow J_X(X)$ is closed

$\Rightarrow \exists \; F \in X^{***}$ s.t. $F|_{J_X(X)} = 0$ and $F(\varphi) \neq 0$.

5/ Since X^* is reflexive, there exists a unique $f \in X^*$ s.t.

$F = \mathcal{J}_{X^*} f$. Hence, for every $x \in X$,

$$0 = F(\mathcal{J}_X x) = (\mathcal{J}_{X^*} f)(\mathcal{J}_X x) = (\mathcal{J}_X x)(f) = f(x)$$

$$\Downarrow$$

$$f = 0 \implies F = 0 \quad ↯$$

because $F(\varphi) \neq 0$!

(3) Let X, Y be normed spaces, $T: X \to Y$ linear. The following are equivalent:

a) The graph $\Gamma(T)$ is closed

b) If $(x_n)_{n \in \mathbb{N}} \subset X$ s.t. $x_n \to 0$ and $(Tx_n)_{n \in \mathbb{N}}$ converges, then $\lim_n Tx_n = 0$

$(a) \Rightarrow (b)$ $\quad \underset{\Gamma(T)}{\underset{\Uparrow}{(x_n, Tx_n)}} \xrightarrow[n \to \infty]{} (0, y)$. Since $\Gamma(T)$ is closed,

$y := \lim_n Tx_n$

then $(0, y) \in \Gamma(T)$ i.e. $y = T(0)$.

Since T is linear $\Rightarrow$ $y = T(0) = 0$ ✓

$(b) \Rightarrow (a)$

$\Gamma(T)$ is closed iff every convergent sequence in $\Gamma(T)$ has its limit in $\Gamma(T)$.

Let $\underset{\Gamma(T)}{\underset{\Uparrow}{(x_n, Tx_n)}} \xrightarrow[n \to \infty]{} (x, y)$. We have to show that $(x, y) \in \Gamma(T)$

$$\underset{\Gamma(T)}{\underset{\Uparrow}{(x_n - x, T(x_n - x))}} \xrightarrow[n \to \infty]{} (0, y - T(x)) \xRightarrow{(b)} y - T(x) = 0 \implies$$

$(x, y) = (x, T(x)) \in \Gamma(T)$

I/(4)

The weak topology is generated by the sets

$$\mathcal{U}(c,f,\varepsilon) := f^{-1}(B(c,\varepsilon)) = \{x \in X : |f(x) - c| < \varepsilon\} \qquad \begin{array}{l} c \in \mathbb{K} \\ f \in X^* \\ \varepsilon > 0. \end{array}$$

$B(c,\varepsilon)$ is open and f is continuous w.r.t. the weak top

$$\Downarrow$$

$$\mathcal{U}(c,f,\varepsilon) \in \sigma(X,X^*) \text{ for every } \begin{array}{l} c \in \mathbb{K} \\ \varepsilon > 0 \\ f \in X^* \end{array}$$

$\Rightarrow$ the topology $\mathcal{J}$ generated by the sets $\mathcal{U}(c,f,\varepsilon)$ satisfies:

$$\mathcal{J} \subseteq \sigma(X,X^*)$$

On the other hand, every open set $\mathcal{U} \subseteq \mathbb{K}$ can be written as

$$\mathcal{U} = \bigcup_{c \in \mathcal{U}} B(c, \varepsilon_c) \quad \text{with some } \varepsilon_c > 0$$

$$\Rightarrow f^{-1}(\mathcal{U}) = \bigcup_{c \in \mathcal{U}} f^{-1}(B(c,\varepsilon_c)) = \bigcup_{c \in \mathcal{U}} \mathcal{U}(c,f,\varepsilon).$$

Thus $f^{-1}(\mathcal{U}) \in \mathcal{J}$ for any open set $\mathcal{U} \in \mathbb{K}$ and thus f is continuous w.r.t. $\mathcal{J}$

$$\Downarrow$$

$$\mathcal{J} \supseteq \sigma(X,X^*)$$

$\circ$!

6/ (5) Let X be an infinite dimensional normed space

$S_X = \{x \in X : \|x\| = 1\}$, then

$$\overline{\{x \in X : \|x\| = 1\}}^{\sigma(X,X^*)} = \{x \in X : \|x\| \leq 1\}$$

Let $\|x\| > 1$. By the Hahn-Banach theorem $\exists f \in X^*$ s.t. $\|f\| = 1$ and $f(x) = \|x\|$.

$\hookrightarrow$ $\{y \in X : |f(y)| > \frac{1+\|x\|}{2}\}$ is a weak-open set,

which
- contains x $\left(|f(x)| = \|x\| \underset{\|x\|>1}{>} \frac{1+\|x\|}{2} \right)$
- disjoint from the unit sphere:

$\left(\text{for } z \in S_X : |f(z)| \underset{\|f\|=1}{\leq} \|z\| \leq 1 \right)$

$\Rightarrow$ <u>x is not in the weak-closure of S_X</u>

On the other hand, let $\|x\| < 1$, and let $U \in \sigma(X,X^*)$ be a weak open set that contains x

<u>We proved</u>: if x is in the weak interior of a set U, then there exists a non-zero vector $z \in X$ s.t. $x + cz \in U$ $\forall c \in \mathbb{K}$

(U is unbounded!)

$\Rightarrow \exists \underset{\neq 0}{z} \in X$ s.t. $x + cz \in U$ $\forall c \in \mathbb{K}$.

Let $g(t) := \|x + tz\|$, $t \in \mathbb{R}$

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$g(t) = \|x + tz\| \quad t \in \mathbb{R} \implies$ • g is continuous

• $g(0) = \|x\| < 1$

• $g(t) \geq |t| \cdot \|z\| - \|x\|$

by the triangle inequ.

$\Downarrow$

$\lim_{t \to \infty} g(t) = \infty$

$\Downarrow$ Bolzano-Weierstrass

$\exists\, t \in \mathbb{R}$ s.t. $x + tz \in \mathcal{U}$ and $\|x + tz\| = 1$

$\Downarrow$

$x + tz \in \mathcal{U} \cap S_X$

$\implies$ every open neighbourhood of x intersects S_X

$\Downarrow$

x is in the closure of S_X

$$\implies \overline{\{x \in X : \|x\| = 1\}}^{\sigma(X, X^*)} = \{x \in X : \|x\| \leq 1\}$$

Remark. Since S_X is closed in the norm topology, but not in the weak topology, the two topologies have to be different.

⑥ $(X, \|.\|_X), (Y, \|.\|_Y)$ normed spaces, $T : X \to Y$ linear. Then

a) T is continuous

$\Downarrow$

b) $\forall\, (x_n)_{n \in \mathbb{N}} \subset X,\ x_n \xrightarrow{w} x \implies Tx_n \xrightarrow{w} Tx$

8/ a) $\Rightarrow$ b) $(x_n)_{n\in\mathbb{N}} \subset X$ s.t. $x_n \xrightarrow{w} x$ for some $x \in X$

Let $f \in Y^*$. If $T: X \to Y$ is continuous, then

$f \circ T \in X^*$ and the weak convergence of $(x_n)_{n\in\mathbb{N}}$

implies

$$\lim_{n\to\infty} (f\circ T)(x_n) = \lim_{n\to\infty} f(Tx_n) \underset{\substack{\uparrow \\ f,T \text{ are continuous}}}{=} (f\circ T)(\lim_{n\to\infty} x_n) = f(Tx)$$

$$\overset{f \text{ was arbitrary}}{\Rightarrow} \quad Tx_n \xrightarrow{w} Tx$$

b) $\Rightarrow$ a) If the linear op. $T: X \to Y$ is not continuous, then there exists $(x_n)_{n\in\mathbb{N}} \subset X$ s.t. $\|x_n\|_X \le 1$ and $\|Tx_n\|_Y \ge n^2 \quad \forall n \in \mathbb{N}$

as $\frac{x_n}{n} \xrightarrow{\|.\|_X} 0 \Rightarrow \frac{x_n}{n} \xrightarrow{w} 0$ but $\left(T(\frac{1}{n}x_n)\right)_{n\in\mathbb{N}}$

is unbounded in Y

and therefore cannot be weakly convergent . !

(7) $(X,\|.\|_X), (Y,\|.\|_Y)$ Banach spaces, $T: \underset{\cap\, X}{D(T)} \to Y$ linear op. with closed graph.

Then

a) T is injective and $\mathrm{Ran}(T) = T(D(T))$ is closed in Y

$\Updownarrow$

b) $\exists\, C > 0$ s.t. $\forall x \in D(T): \|x\|_X \le C\|Tx\|_Y$.

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a) $\Rightarrow$ b) As a closed subspace of a complete space is complete

$$\Downarrow$$

$(Ran(T), \|.\|_Y)$ is complete

$T: \underset{\substack{\cap \\ X}}{D(T)} \to Ran(T)$ is bijective with closed graph

+ $Ran(T)$ is Banach space

$\Downarrow$ Inverse mapping theorem

$\exists\ T^{-1}: Ran(T) \to D(T)$ <u>bounded</u>

$\|T^{-1}\| =: C < \infty$, $\forall x \in D(T)$

$$\|x\|_X = \|T^{-1}Tx\|_X \leq \|T^{-1}\| \cdot \|Tx\|_Y = C\|Tx\|_Y$$ ✓

b) $\Rightarrow$ a) Let $x \in D(T)$ with $Tx = 0 \overset{(b)}{\Rightarrow} \|x\|_X \leq 0 \Rightarrow \underline{x = 0}$

$$\Downarrow$$

<u>T is injective</u>

Let $(y_n)_{n\in\mathbb{N}} \subset Ran(T)$ converging to some $y \in Y$.

$$\Downarrow$$

$\exists\ (x_n)_{n\in\mathbb{N}} \in D(T)$ st $Tx_n = y_n$

$\hookrightarrow$
$\forall m,n \in \mathbb{N}$ $\quad \|x_n - x_m\|_X \leq C\|Tx_n - Tx_m\|_Y = C\|y_n - y_m\|_Y$

$(y_n)_{n\in\mathbb{N}}$ is Cauchy $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X.

As X is complete, $\exists\ \underset{\substack{\cap \\ X}}{x} = \lim_{n\to\infty} x_n$.

10)

Since the graph of T is assumed to be closed

$$\Downarrow$$

$$x \in D(T) \text{ and } Tx = y$$

$\Rightarrow$ $y \in Ran(T)$ $\Rightarrow$ $Ran(T)$ is a closed subspace of Y.