Functional Analysis,
Exercises 6.

1. Show that a norm $\|.\|$ in a complex vector space is induced by an inner product if and only if it satisfies the Parallelogram Identity, i.e. iff

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

holds for all vectors x and y. Moreover, show that if $\|.\|$ satisfies the Parallelogram Identity, then the inner product $\langle\, .\,,.\,\rangle$ that induces $\|.\|$ is given by

$$\langle\, x, y\,\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \mathrm{i}\,\|x+\mathrm{i}\,y\|^2 - \mathrm{i}\,\|x-\mathrm{i}\,y\|^2\right).$$

2. Prove the Appolonius theorem, i.e. in an inner product space for all x, y, z vectors the identity

$$\|x-y\|^2 + \|x-z\|^2 = 2\left(\|x - \frac{y+z}{2}\|^2 + \|\frac{y-z}{2}\|^2\right)$$

holds.

3. Let X be an inner product space. Prove that the following statements are equivalent.

   (a) $x \perp y$
   (b) for all scalars λ, $\|x+\lambda y\| = \|x-\lambda y\|$
   (c) for all scalars λ, $\|x+\lambda y\| \geq \|x\|$.

4. Given x, y in a complex inner product space X, prove that

$$\langle\, x, y\,\rangle = \frac{1}{2\pi}\int_0^{2\pi} \|x + e^{\mathrm{i}\,t} y\|^2 e^{\mathrm{i}\,t}\,\mathrm{d}\,t.$$

5. Let X be an inner product space and $x_1, \ldots, x_{100}$ unit vectors in X such that $|\langle\, x_n, x_m\,\rangle| = 1/10$ for all $n \neq m$. Find a sharp estimate for $\|x_1 + \cdots + x_{100}\|$.

6. If x is a vector in an inner product space, the show that

$$\|x\| = \sup_{\|y\|=1} |\langle x, y \rangle|.$$

7. Assume that a sequence $(x_n)_{n\in\mathbb{N}}$ in a Hilbert space satisfies $\langle x_n, x \rangle \to \|x\|^2$ and $\|x_n\| \to \|x\|$ as $n \to \infty$. Show that $x_n \to x$.

8. Show that the norms of the following Banach spaces cannot be induced by inner products.

   (a) The norm $\|x\| = \max\{|x_1|, \ldots, \|x_n\|\}$ on $\mathbb{R}^n$.

   (b) The sup norm on $C[a, b]$

   (c) The $\|.\|_p$ norm on any $L^p(\mu)$ space for each $1 \le p \le \infty$ with $p \neq 2$.

9. Recall that for the vectors $x_1, \ldots, x_n$ in a Hilbert space $\mathcal{H}$ the $n \times n$ Gram matrix $G(x_1, \ldots, x_n)$ whose i, j entry is $\langle x_i, x_j \rangle$ we called the Gram matrix of the given vectors. Prove the following statements.

   (a) Every Gram matrix is positive semidefinite. It is positive definite iff the vectors x_j are linearly independent.

   (b) Every positive semidefinite matrix is a Gram matrix.

   (c) Let $x_1, \ldots, x_n$ be linearly independent vectors in $\mathcal{H}$ and let $\mathcal{M}$ be their linear span. The for every $y \in \mathcal{H}$

$$\operatorname{dist}(y, \mathcal{M}) = \left(\frac{\det G(y, x_1, \ldots, x_n)}{\det G(x_1, \ldots, x_n)} \right)^{1/2}.$$

10. If $(x_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$, then show that $\lim_n \langle x_n, y \rangle = 0$ for each $y \in \mathcal{H}$.

11. Let $(\phi_n)_{n\in\mathbb{N}}$ be an orthonormal sequence of functions in the Hilbert space $L^2[-1, 1]$. Show that the the sequence of functions $(\psi_n)_{n\in\mathbb{N}}$, where

$$\psi_n(x) = \left(\frac{2}{b-a}\right)^{1/2} \phi_n \left(\frac{2}{b-a}(x - \frac{b+a}{2}) \right)$$

is an orthonormal sequence in the Hilbert space $L^2[a, b]$.

12. Find $a, b, c \in \mathbb{C}$ which minimize the value of the integral

$$\int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \,\mathrm{d}\,x.$$