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Functional Analysis - Exercises 6.
Solutions

(1) A norm $\|.\|$ is induced by an inner product iff

$$\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2) \qquad \text{(Parallelogram identity)}$$

If $\|.\|$ is induced by the inner product $\langle .,. \rangle$, then $\forall x,y \in X$ (inner product space)

$$\|x+y\|^2+\|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle =$$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$
$$+ \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle = 2(\langle x,x\rangle + \langle y,y\rangle) =$$
$$= 2(\|x\|^2+\|y\|^2) \checkmark$$

Now assume that the norm on X is such that the parallelogram identity holds and for $x,y \in X$ define $\langle x,y\rangle$ by the polarization formula:

$$\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

- positivity:

$$4\langle x,x\rangle = \|x+x\|^2 - \|x-x\|^2 + i\|x+ix\|^2 - i\|\underbrace{x-ix}_{-i(x+ix)}\|^2 = \quad \text{and } |-i|=1$$
$$= 4\|x\|^2 + i\|x+ix\|^2 - i\|x+ix\|^2 = 4\|x\|^2 \geq 0 \checkmark$$

- hermiticity:

$$4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|\overbrace{x+iy}^{i(y-ix)}\|^2 - i\|\overbrace{x-iy}^{-i(y+ix)}\|^2 =$$
$$= \|y+x\|^2 - \|y-x\|^2 + i\|-ix+y\|^2 - i\|ix+y\|^2 = 4\,\overline{\langle y,x\rangle} \checkmark$$

2)

- <u>additivity</u>

$$4\left[\langle x,y\rangle+\langle x,z\rangle\right] = \|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2+$$
$$+\|x+z\|^2-\|x-z\|^2+i\|x+iz\|^2-i\|x-iz\|^2=$$

$$= \left\|x+\frac{y+z}{2}+\frac{y-z}{2}\right\|^2-\left\|x-\frac{y+z}{2}-\frac{y-z}{2}\right\|^2+\left\|x+\frac{y+z}{2}-\frac{y-z}{2}\right\|^2-\left\|x-\frac{y+z}{2}+\frac{y-z}{2}\right\|^2$$
$$+i\left\|x+i\frac{y+z}{2}+i\frac{y-z}{2}\right\|^2-i\left\|x-i\frac{y+z}{2}-i\frac{y-z}{2}\right\|^2+i\left\|x+i\frac{y+z}{2}-i\frac{y-z}{2}\right\|^2-i\left\|x-i\frac{y+z}{2}+i\frac{y-z}{2}\right\|^2$$

$$= 2\left\|x+\frac{y+z}{2}\right\|^2+2\left\|\frac{y-z}{2}\right\|^2-2\left\|x-\frac{y+z}{2}\right\|^2-2\left\|\frac{y-z}{2}\right\|^2=$$

parallelogram identity for the underlined pairs

$$+2i\left\|x+i\frac{y+z}{2}\right\|^2+2i\left\|\frac{y-z}{2}\right\|^2-2i\left\|x-i\frac{y+z}{2}\right\|^2-2i\left\|\frac{y-z}{2}\right\|^2$$

$$= 2\left\|x+\frac{y+z}{2}\right\|^2-2\left\|x-\frac{y+z}{2}\right\|^2+2i\left\|x+i\frac{y+z}{2}\right\|^2-2i\left\|x-i\frac{y+z}{2}\right\|^2$$
$$= 2\cdot 4\left\langle x,\frac{y+z}{2}\right\rangle$$

If we choose $z=0$ we find $\langle x,y\rangle = 2\langle x,\frac{y}{2}\rangle$, hence

$$\langle x,y\rangle+\langle x,z\rangle = 2\left\langle x,\frac{y+z}{2}\right\rangle = \langle x,y+z\rangle \quad ✓$$

- <u>homogeneity</u>

From the additivity we obtain $\langle \lambda x,y\rangle = \lambda\langle x,y\rangle$ for all $\lambda\in\mathbb{Q}$

Since $\langle ix,y\rangle = i\langle x,y\rangle \Rightarrow$ homogeneity is proved for $\lambda = \mathbb{Q}+i\mathbb{Q}$

For fixed x,y the functions $\lambda\mapsto\lambda\langle x,y\rangle$ and $\lambda\mapsto\langle\lambda x,y\rangle$ are continuous and must be equal because they are equal on the dense subset $\mathbb{Q}+i\mathbb{Q}$ of $\mathbb{C}$. .!

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② Prove the Appolonius theorem, ie

$$\|x-y\|^2+\|x-z\|^2=2\left(\|x-\tfrac{y+z}{2}\|^2+\|\tfrac{y-z}{2}\|^2\right)$$

By the parallelogram rule:

$$2\left(\|x-\tfrac{y+z}{2}\|^2+\|\tfrac{y-z}{2}\|^2\right)=\|x-\tfrac{y+z}{2}+\tfrac{y-z}{2}\|^2+\|x-\tfrac{y+z}{2}-\tfrac{y-z}{2}\|^2=$$

$$=\|x-\frac{y+z-y+z}{2}\|^2+\|x-\frac{y+z+y-z}{2}\|^2=$$

$$=\|x-z\|^2+\|x-y\|^2 \quad !$$

③ $(X,\langle .,.\rangle)$ inner product space. Equivalent statements

a) $x\perp y$

b) $\forall\lambda\in\mathbb{K}\quad \|x+\lambda y\|=\|x-\lambda y\|$

c) $\forall\lambda\in\mathbb{K}\quad \|x+\lambda y\|\geq\|x\|$.

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a ⇒ b) $\|x\pm\lambda y\|^2=\langle x\pm\lambda y,x\pm\lambda y\rangle=\|x\|^2\pm\lambda\langle y,x\rangle\pm\bar\lambda\langle x,y\rangle+|\lambda|^2\cdot\|y\|^2=$

$$\underset{\langle x,y\rangle=0}{=}\|x\|^2+|\lambda|^2\|y\|^2$$

b ⇒ c) $\|x+\lambda y\|^2\underset{b)}{=}\frac{1}{2}\left(\|x+\lambda y\|^2+\|x-\lambda y\|^2\right)=\|x\|^2+|\lambda|^2\cdot\|y\|^2\geq\|x\|^2$

c) ⇒ a) $\forall\lambda\in\mathbb{K}\quad \|x+\lambda y\|^2-\|x\|^2=\lambda\underbrace{\langle y,x\rangle}_{\overline{\langle x,y\rangle}}+\bar\lambda\langle x,y\rangle+|\lambda|^2\|y\|^2=$

$$=2\,\mathrm{Re}\{\bar\lambda\langle x,y\rangle\}+|\lambda|^2\cdot\|y\|^2\geq 0$$

for $\lambda:=-\frac{\langle x,y\rangle}{\|y\|^2}$ we have $-2\frac{|\langle x,y\rangle|^2}{\|y\|^2}+\frac{|\langle x,y\rangle|^2}{\|y\|^2}=-\frac{|\langle x,y\rangle|^2}{\|y\|^2}\geq 0 \iff \langle x,y\rangle=0$!

4)

(S) In $(X,\langle .,. \rangle)$ $\quad \langle x,y \rangle = \frac{1}{2\pi}\int_0^{2\pi} \|x+e^{it}y\|^2 e^{it}\,dt$

$$\|x+e^{it}y\|^2 e^{it} = \langle x+e^{it}y, x+e^{it}y\rangle e^{it} = \left(\|x\|^2 + e^{it}\langle y,x\rangle + e^{-it}\langle x,y\rangle + \|y\|^2\right)e^{it}$$

$$= (\|x\|^2+\|y\|^2)e^{it} + \langle x,y\rangle + \langle y,x\rangle e^{i2t}$$

$$\int_0^{2\pi} e^{it}\,dt = \left[\frac{e^{it}}{i}\right]_0^{2\pi} = 0 \,, \quad \int_0^{2\pi} e^{i2t}\,dt = \left[\frac{e^{i2t}}{2i}\right]_0^{2\pi} = 0$$

$$\Rightarrow \langle x,y\rangle = \frac{1}{2\pi}\int_0^{2\pi} \|x+e^{it}y\|^2 e^{it}\,dt \,.$$

ü!

(J) $(X,\langle .,. \rangle)$ inner product space, $x_1,\dots,x_{100}$ unit vectors in X s.t. $|\langle x_n, x_m\rangle| = \frac{1}{10}$ if $n\neq m$. Estimate $\|x_1+\dots+x_{100}\|$.

$$\left\|\sum_{n=1}^{100} x_n\right\|^2 = \left\langle \sum_{n=1}^{100} x_n, \sum_{k=1}^{100} x_k\right\rangle = \sum_{n,k=1}^{100}\langle x_n, x_k\rangle = \sum_{n=1}^{100}\|x_n\|^2 +$$

$$+ \sum_{n\neq m}^{100}\langle x_n,x_m\rangle \leq 100 + 9900\cdot\frac{1}{10} = \underline{1090}$$

$$\uparrow$$

$2\cdot\binom{100}{2} = 100\cdot 99 = 9900$ ordered pairs can be selected from 100 elements

$$\Rightarrow \boxed{\left\|\sum_{n=1}^{100} x_n\right\| \leq \sqrt{1090}}$$

To see that the estimate is sharp consider the sequence $x_n \in \ell_2$:

~~$x_1 = (\sqrt{0{,}1}, 0, 0, \dots, \sqrt{0{,}9},$~~

$x_1 = (\sqrt{0{,}1}, \sqrt{0{,}9}, 0, 0, \dots)$

$x_2 = (\sqrt{0{,}1}, 0, \sqrt{0{,}9}, 0, 0, \dots)$

$x_3 = (\sqrt{0{,}1}, 0, 0, \sqrt{0{,}9}, 0, \dots)$

$\vdots$

5)

then $\|x_n\|_2 = 1 \quad \forall n = 1, \dots, 100$

$$\sum_{n=1}^{100} x_n = \big(100 \cdot \sqrt{0{,}1}, \sqrt{0{,}5}, \sqrt{0{,}5}, \dots, \underset{\substack{\uparrow \\ 101}}{\sqrt{0{,}5}}, 0, 0, 0 \dots \big)$$

$$\Downarrow$$

$$\Big\|\sum_{n=1}^{\infty} x_n\Big\|_2 = \sqrt{1050}$$

. !

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$$\|x\| = \sup_{\|y\|=1} |\langle x, y\rangle|$$

If $x = 0 \Rightarrow$ it is OK.

For $x \neq 0$, if $\|y\| = 1$, then by the Cauchy-Schwarz:

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\| = \|x\|$$

$$\hookrightarrow \quad \sup_{\|y\|=1} |\langle x, y\rangle| \leq \|x\|$$

For the reverse, let $z = \frac{x}{\|x\|} \rightsquigarrow \|z\| = 1$

$$\sup_{\|y\|=1} |\langle x, y\rangle| \geq |\langle x, z\rangle| = |\langle x, \tfrac{x}{\|x\|}\rangle| = \frac{|\langle x, x\rangle|}{\|x\|} = \|x\|$$

$$\Rightarrow \quad \|x\| = \sup_{\|y\|=1} |\langle x, y\rangle|$$

. !

⑦ $(x_n)_{n\in\mathbb{N}} \subset \mathcal{H}$. $\langle x_n, x\rangle \to \|x\|^2$ and $\|x_n\| \to \|x\|$, then

$$x_n \to x$$

$$\langle x_n, x\rangle \to \|x\|^2 \implies \langle x, x_n\rangle = \overline{\langle x_n, x\rangle} \to \overline{\|x\|^2} = \|x\|^2$$

$$\hookrightarrow \|x_n - x\|^2 = \langle x_n - x, x_n - x\rangle = \|x_n\|^2 - \langle x, x_n\rangle - \langle x_n, x\rangle + \|x\|^2 \xrightarrow[n\to\infty]{}$$

$$\to \|x\|^2 - \|x\|^2 - \|x\|^2 + \|x\|^2 = 0$$

so $\|x_n - x\| \to 0$ i.e. $x_n \to x$!

⑧ The norms of the following Banach spaces cannot be induced by inner products:

a) $\mathbb{R}^n$ with $\|x\| = \max\{|x_1|, \dots, |x_n|\}$

we have to check the Parallelogram Identity:

for $x = (1, 0, \dots, 0)$ and $y = (0, 1, 0, \dots 0)$

$$\|x\| = \|y\| = \|x+y\| = \|x-y\| = 1$$

$$\hookrightarrow \|x+y\|^2 + \|x-y\|^2 = 2 \neq 2(\|x\|^2 + \|y\|^2) = 4$$ ✓

b) $C[a,b]$ with sup norm

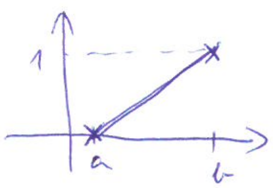
$g(x) := 1 \quad x \in [a,b]$

$f(x) = \dfrac{x-a}{b-a}$

$\implies \|g\|_\infty = \|f\|_\infty = 1$

$\|f+g\|_\infty = 2$

$\|f-g\|_\infty = 1$



$$\hookrightarrow (\|f+g\|_\infty + \|f-g\|_\infty) = 5 \neq 2(\|f\|_\infty^2 + \|g\|_\infty^2) = 4$$

7/ c) $L^p(\mu)$ with $p \in [1,\infty]$ $p \neq 2$

Assume E and F are disjoint measurable sets s.t.

$$0 < \mu^*(E) < \infty \quad \text{and} \quad 0 < \mu^*(F) < \infty$$ μ^* is the outer measure

$$\Rightarrow \mu^*(E) = \mu(E)$$
$$\mu^*(F) = \mu(F)$$

- $p = \infty$ case χ_E: the characteristic function of E

$$\|\chi_E\|_\infty = \|\chi_F\|_\infty = 1 \quad \text{and} \quad \|\chi_E + \chi_F\|_\infty = \|\chi_E - \chi_F\|_\infty = 1$$

$$\hookrightarrow \|\chi_E + \chi_F\|_\infty^2 + \|\chi_E - \chi_F\|_\infty^2 = 2 \neq 2(\|\chi_E\|_\infty^2 + \|\chi_F\|_\infty^2) = 4$$

- $1 \leq p < \infty$ with $p \neq 2$

$$f := [\mu^*(E)]^{-1/p} \cdot \chi_E \quad , \quad g := [\mu^*(F)]^{-1/p} \cdot \chi_F$$

$$\Rightarrow \|f\|_p^p = \int_X |f|^p \, d\mu = \frac{1}{\mu(E)} \int_X \chi_E \, d\mu = \frac{1}{\mu(E)} \int_E d\mu = 1$$

similarly $\|g\|_p = 1$

$$\Rightarrow 2\left[\|f\|_p^2 + \|g\|_p^2\right] = 4 = 2^2$$

$$|f+g|^p = |f-g|^p = |f|^p + |g|^p$$

$$\hookrightarrow \|f+g\|_p^2 + \|f-g\|_p^2 \overset{\downarrow}{=} \left[\left(\int |f|^p d\mu + \int |g|^p d\mu\right)^{1/p}\right]^2 + \left[\left(\int |f|^p d\mu + \int |g|^p d\mu\right)^{1/p}\right]^2 =$$

$$= (2^{1/p})^2 + (2^{1/p})^2 = 2 \cdot 2^{2/p} = 2^{1+\frac{2}{p}} \underset{\text{if } p \neq 2}{\neq} 2^2 \quad . !$$

(9) $[G(x_1,\dots,x_n)]_{ij} = \langle x_i, x_j\rangle$ Gram matrix.

a) G is positive semidefinite. ~~Hmm~~

G is positive semidefinite iff $\forall\, u \in \mathbb{K}^n$ $\langle Gu, u\rangle \geq 0$.

Let $u = (u_1, u_2, \dots, u_n) \in \mathbb{K}^n$

$$\langle Gu, u\rangle = \sum_{i=1}^{n} (Gu)_i \overline{u_i} = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} G_{ij} u_j\Big)\overline{u_i} =$$

$$= \sum_{i,j=1}^{n} \langle x_i, x_j\rangle \overline{u_i} u_j = \Big\langle \sum_{i=1}^{n} u_i x_i, \sum_{j=1}^{n} u_j x_j \Big\rangle =$$

$$= \Big\| \sum_{i=1}^{n} u_i x_i \Big\|^2 \geq 0$$ ✓

b) Every positive semidefinite matrix is a Gram matrix.

If A is positive semidefinite, then ~~for an~~ it has a square root $A^{1/2}$ and for the ONB $\{e_1, \dots, e_n\}$:

$$a_{ij} = \langle Ae_i, e_j\rangle = \langle A^{1/2} e_i, A^{1/2} e_j\rangle = \langle x_i, x_j\rangle$$

for $x_i = A^{1/2} e_i$ and $x_j = A^{1/2} e_j$

!

c) $x_1, x_2, \dots, x_n$ linearly independent

$\mathcal{M} := \operatorname{span}\{x_1, \dots, x_n\}$

$\forall y \in \mathcal{H}$:

$$\operatorname{dist}(y, \mathcal{M}) = \left(\frac{\det G(y, x_1, \dots, x_n)}{\det G(x_1, \dots, x_n)}\right)^{1/2}$$

a)

We know that there exists a unique $w = \sum_{i=1}^{n} \alpha_i x_i \in \mathcal{M}$

st $\|y - w\| = \mathrm{dist}(y, \mathcal{M})$ and $y - w \perp \mathcal{M}$.

$\hookrightarrow 0 = \langle y - w, x_j \rangle = \langle y, x_j \rangle - \sum_{i=1}^{n} \alpha_i \langle x_i, x_j \rangle \qquad 1 \leq j \leq n$

$\Downarrow$ we have a system of linear equations

$$(*) \begin{cases} \alpha_1 \langle x_1, x_1 \rangle + \alpha_2 \langle x_2, x_1 \rangle + \ldots + \alpha_n \langle x_n, x_1 \rangle = \langle y, x_1 \rangle \\ \vdots \\ \alpha_1 \langle x_1, x_n \rangle + \alpha_2 \langle x_2, x_n \rangle + \ldots + \alpha_n \langle x_n, x_n \rangle = \langle y, x_n \rangle \end{cases}$$

$\Downarrow$

$$\underbrace{\begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_2, x_1 \rangle & \ldots & \langle x_n, x_1 \rangle \\ \vdots & & & \\ \langle x_1, x_n \rangle & \langle x_2, x_n \rangle & \ldots & \langle x_n, x_n \rangle \end{pmatrix}}_{G(x_1, \ldots, x_n)} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \langle y, x_1 \rangle \\ \vdots \\ \langle y, x_n \rangle \end{pmatrix}$$

Since the set of α_i is unique, we have by Cramer's rule:

$$\alpha_j = \frac{D_j}{\det G(x_1, \ldots, x_n)}$$

, where D_j is the determinant of the matrix which is obtained by replacing the jth column of the ~~G~~ Gram matrix by the column of inhomogeneity vector.

10/ Since $w=\sum_{i=1}^{n} \alpha_i x_i$, w can be get as

$$w = -\frac{1}{\det G(x_1,\dots,x_n)} \det \left(\begin{array}{cccc|c} \langle x_1,x_1\rangle & \langle x_2,x_1\rangle & \dots & \langle x_n,x_1\rangle & \langle y,x_1\rangle \\ \vdots & & & & \\ \langle x_1,x_n\rangle & \langle x_2,x_n\rangle & \dots & \langle x_n,x_n\rangle & \langle y,x_n\rangle \\ \hline x_1 & x_2 & \dots & x_n & 0 \end{array}\right)$$

expanding the determinant of the last row.

Let $d=\mathrm{dist}(y,\mathcal{M})$.

Since $y-w \perp \mathcal{M}$ and $w\in\mathcal{M} \rightsquigarrow \langle y-w,w\rangle=0$

$\Rightarrow$ $d^2=\|y-w\|^2=\langle y-w,y-w\rangle=\langle y-w,y\rangle=\langle y,y\rangle-\langle w,y\rangle=$

$$=\langle y,y\rangle-\sum_{i=1}^{n}\alpha_i\langle x_i,y\rangle$$

$\Updownarrow$

$$\alpha_1\langle x_1,y\rangle+\alpha_2\langle x_2,y\rangle+\dots+\alpha_n\langle x_n,y\rangle=\langle y,y\rangle-d^2.$$

Attaching this equation to $(*)$, denoting the unknowns by z_i:

$$\left.\begin{array}{l} \langle x_1,x_1\rangle z_1+\langle x_2,x_1\rangle z_2+\dots+\langle x_n,x_1\rangle z_n+\langle -y,x_1\rangle z_{n+1}=0 \\ \vdots \\ \langle x_1,x_n\rangle z_1+\langle x_2,x_n\rangle z_2+\dots+\langle x_n,x_n\rangle z_n+\langle -y,x_n\rangle z_{n+1}=0 \\ \langle x_1,y\rangle z_1+\langle x_2,y\rangle z_2+\dots+\langle x_n,y\rangle z_n+(d^2-\langle y,y\rangle)z_{n+1}=0 \end{array}\right\}$$

A non-trivial solution as we have seen is

$$z_i=\alpha_i \quad 1\le i\le n, \quad z_{n+1}=1$$

Hence by the Cramer rule:

$$\det \begin{pmatrix} \langle x_1,x_1\rangle & \langle x_2,x_1\rangle & \dots & \langle x_n,x_1\rangle & \langle -y,x_1\rangle \\ \vdots & & & & \vdots \\ \langle x_1,x_n\rangle & \langle x_2,x_n\rangle & \dots & \langle x_n,x_n\rangle & \langle -y,x_n\rangle \\ \langle x_1,y\rangle & \langle x_2,y\rangle & \dots & \langle x_n,y\rangle & d^2-\langle y,y\rangle \end{pmatrix} = 0$$

Expanding the determinant by using the last column we get the statement.

(10) If $(x_n)_{n\in\mathbb{N}}$ is an ON sequence in $\mathcal{H}$, then $\lim_{n\to\infty}\langle x_n,y\rangle = 0$ $\forall y\in\mathcal{H}$

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from the Bessel's inequality we have:

$$\sum_{n=1}^{\infty} |\langle x_n,y\rangle|^2 \le \|y\|^2 < \infty$$

$$\Rightarrow \quad |\langle x_n,y\rangle|^2 \xrightarrow[n\to\infty]{} 0 \quad \Rightarrow \quad \langle x_n,y\rangle \xrightarrow[n\to\infty]{} 0 \qquad .!$$

(11) $(\varphi_n)_{n\in\mathbb{N}}$ is an ON sequence in $L^2[-1,1]$

$$\Rightarrow \quad \psi_n(x) := \left(\frac{2}{b-a}\right)^{1/2} \varphi_n\left(\frac{2}{b-a}\left(x-\frac{b+a}{2}\right)\right)$$

$\hookrightarrow$ $(\psi_n)_{n\in\mathbb{N}}$ is ON sequence in $L^2[a,b]$.

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$$\langle \psi_n, \psi_m \rangle = \int_a^b \psi_n(x) \overline{\psi_m(x)}\, dx =$$

$$= \frac{2}{b-a} \int_a^b \varphi_n\left(\frac{2}{b-a}\left(x - \frac{b+a}{2}\right)\right) \overline{\varphi_m}\left(\frac{2}{b-a}\left(x-\frac{b+a}{2}\right)\right) dx \;\; \stackrel{}{=}$$

substitute $\quad t := \frac{2}{b-a}\left(x - \frac{b+a}{2}\right)$

$\hookrightarrow \quad dt = \frac{2}{b-a}\, dx$

if $\quad a \leq x \leq b \quad \Rightarrow \quad \frac{2}{b-a}\underbrace{\left(a - \frac{b+a}{2}\right)}_{\frac{a-b}{2}} \leq t \leq \frac{2}{b-a}\underbrace{\left(b - \frac{b+a}{2}\right)}_{\frac{b-a}{2}}$

$\hookrightarrow \quad -1 \leq t \leq 1$

$$= \int_{-1}^{1} \varphi_n(t) \overline{\varphi_m}(t)\, dt = \delta_{nm}$$

$\circ$!

12) Find $a, b, c \in \mathbb{C}$ which minimizes

$$\int_{-1}^{1} |x^3 - a - bx - cx^2|^2\, dx$$

.

13)

We have seen : if $x_1, \dots, x_n$ is an <u>orthonormal set</u> in $\mathcal{H}$ then given $x \in \mathcal{H}$

$$\|x - \sum_{i=1}^{n} \alpha_i x_i\| \text{ is minimal iff } \alpha_i = \langle x, x_i \rangle \quad i = 1, \dots n$$

here
$$\int_{-1}^{1} |x^3 - (a+bx+cx^2)|^2 dx = \|x^3 - (a+bx+cx^2)\|_2^2$$
in $L^2[-1,1]$

let $y_1 = 1$, $y_2 = x$, $y_3 = x^2$ $\Rightarrow$ linearly independent vectors

$\Downarrow$ Gram-Schmidt process

$$w_1 = y_1 = 1 \quad \rightsquigarrow \quad \|w_1\|^2 = \int_{-1}^{1} 1\, dx = 2 \quad \rightsquigarrow \quad x_1 := \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}$$

$$w_2 = y_2 - \langle y_2, x_1 \rangle x_1 \underset{\uparrow}{=} y_2 = x$$

$$\langle y_2, x_1 \rangle = \int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}} dx = \frac{1}{\sqrt{2}} \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

$$\|w_2\|^2 = \int_{-1}^{1} |x|^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \quad \rightsquigarrow \quad \|w_2\| = \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow x_2 = \frac{w_2}{\|w_2\|} = \frac{\sqrt{3}\,x}{\sqrt{2}}$$

$$w_3 = y_3 - \langle y_3, x_1 \rangle x_1 - \langle y_3, x_2 \rangle x_2 \underset{\uparrow}{=} x^2 - \frac{\sqrt{2}}{3} \frac{1}{\sqrt{2}} = x^2 - \frac{1}{3}$$

$$\langle y_3, x_1 \rangle = \int_{-1}^{1} x^2 \cdot \frac{1}{\sqrt{2}} dx = \frac{1}{\sqrt{2}} \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{\sqrt{2}} \cdot \frac{2}{3} = \frac{\sqrt{2}}{3}$$

$$\langle y_3, x_2 \rangle = \int_{-1}^{1} x^2 \cdot \frac{\sqrt{3}}{\sqrt{2}} x\, dx = \frac{\sqrt{3}}{\sqrt{2}} \left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

$$\|w_3\|^2 = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx = \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx =$$

$$= \left[\frac{x^5}{5} - \frac{2}{9}x^3 + \frac{1}{9}x\right]_{-1}^{1} = 2\left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right) = \frac{8}{45}$$

$$\Rightarrow \|w_3\| = \sqrt{\frac{8}{45}} \qquad \Rightarrow \quad x_3 = \frac{w_3}{\|w_3\|} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

$$\Rightarrow \quad x_1 = \frac{1}{\sqrt{2}}, \quad x_2 = \frac{\sqrt{3}}{\sqrt{2}}x, \quad x_3 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \text{ is}$$

an ONB in the subspace spanned by $\{1, x, x^2\}$

$$\Rightarrow \quad \|x^3 - (a + bx + cx^2)\| = \cancel{\|x^3 - (a\sqrt{2}(\tfrac{1}{\sqrt{2}})}$$

$$a + bx + cx^2 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 =$$

$$= \alpha_1 \frac{1}{\sqrt{2}} + \alpha_2 \frac{\sqrt{3}}{2}x + \alpha_3 \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

$$= \left(\alpha_1 \frac{1}{\sqrt{2}} - \alpha_3 \sqrt{\frac{45}{8}} \cdot \frac{1}{3}\right) + \alpha_2 \frac{\sqrt{3}}{2}x + \alpha_3 \sqrt{\frac{45}{8}}x^2$$

$$\Rightarrow \quad a = \left(\alpha_1 \frac{1}{\sqrt{2}} - \alpha_3 \sqrt{\frac{45}{8}} \cdot \frac{1}{3}\right)$$

$$b = \alpha_2 \cdot \frac{\sqrt{3}}{2}$$

$$c = \alpha_3 \sqrt{\frac{45}{8}}$$

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with this choice

$$\| x^3 - (a + bx + cx^2) \|_2 = \| x^3 - (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3) \|_2$$

it is minimal if

$$\alpha_1 = \langle x^3, x_1 \rangle = \int_{-1}^{1} x^3 \cdot \frac{1}{\sqrt{2}} \, dt = \frac{1}{\sqrt{2}} \left[\frac{x^4}{4} \right]_{-1}^{1} = 0$$

$$\alpha_2 = \langle x^3, x_2 \rangle = \int_{-1}^{1} x^3 \frac{\sqrt{3}}{\sqrt{2}} x \, dt = \frac{\sqrt{3}}{\sqrt{2}} \int_{-1}^{1} x^4 dt = \frac{\sqrt{3}}{\sqrt{2}} \underbrace{\left[\frac{x^5}{5} \right]_{-1}^{1}}_{\frac{2}{5}} =$$

$$= \frac{\sqrt{6}}{5}$$

$$\alpha_3 = \langle x^3, x_3 \rangle = \sqrt{\frac{45}{8}} \int_{-1}^{1} x^3 \left(x^2 - \frac{1}{3} \right) dt = \sqrt{\frac{45}{8}} \int_{-1}^{1} \left(x^5 - \frac{1}{3} x^3 \right) dt =$$

$$= \sqrt{\frac{45}{8}} \left[\frac{x^6}{6} - \frac{x^4}{12} \right]_{-1}^{1} = 0$$

$$\Rightarrow \quad \alpha_1 = \alpha_3 = 0 \quad \Rightarrow \quad a = 0, \; c = 0$$

$$\alpha_2 = \frac{\sqrt{6}}{5} \quad \Rightarrow \quad b = \frac{\sqrt{6}}{5} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{18}}{10}$$