# Functional Analysis,
# Exercises 7.

1. Show that a closed subspace $M$ in a Hilbert space is invariant under a linear bounded operator $A$ iff $M^\perp$ is invariant under $A^*$. Thus $M$ reduces $A$ iff it is invariant under both $A$ and $A^*$.

2. Let $X, Y$ be Banach spaces. Suppose that $T \in \mathcal{B}(X, Y)$ is *bounded below*, that is, there exists $C > 0$ s.t. $\|Tx\| \geq C\|x\|$, for all $x \in X$. Prove that $\operatorname{Ran} T$ is closed.

3. Let $X$ be a Banach space, $T \in \mathcal{B}(X)$.

   (a) Show that $T$ is invertible iff $T$ is bounded below and $\operatorname{Ran} T$ is dense.

   (b) Let $X$ be a Hilbert space. Show that $T$ is invertible iff $T$ is bounded below and $\operatorname{Ker} T^* = \{0\}$.

   (c) Let $X$ be a Hilbert space, $T$ be normal operator. Show that $T$ is invertible iff $T$ is bounded below.

4. Let $X$ be a Banach space, $T \in \mathcal{B}(X)$. Show that $\operatorname{Ran} T$ is not dense iff there exists a nonzero $\varphi \in X^*$ s.t. $\varphi \circ T = 0$.

5. Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$ be normal.

   (a) Show that $\operatorname{Ker} T$ is $T^*$-invariant.

   (b) Show that $(\operatorname{Ker} T)^\perp$ is $T$-invariant.

   (c) Prove that $\operatorname{Ker} T = \operatorname{Ker} T^2$.

   (d) Prove that $\operatorname{Ker} T = \ker T^k$ for any positive integer $k$.

6. Let $T \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator. Show that if $\langle Tx, x \rangle =$ for all $x \in \mathcal{H}$, then $T = 0$.

7. Prove that an operator $T \in \mathcal{B}(\mathcal{H})$ is normal iff $\|Tx\| = \|T^*x\|$ for all $x \in \mathcal{H}$.

8. Let define the following $T : \ell_2 \to \ell_2$ operator by

$$T(x_1, x_2, \dots) = \left( \frac{x_1 + x_2}{2}, \frac{x_2 + x_3}{2}, \frac{x_3 + x_4}{2}, \dots \right).$$

Determine the adjoint operator $T^*$.

9. Let define the following $T : \ell_\infty \to \ell_\infty$ operator by

$$T(x_1, x_2, x_3, x_4, \dots) = \left( x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots \right).$$

Show that $T$ is compact. Derermine its spectrum (point, continous, residual).

10. Let define the following $T : \ell_\infty \to \ell_\infty$ operator by

$$T(x_1, x_2, x_3, x_4, \dots) = \left( x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \frac{x_1 + x_2 + x_3 + x_4}{4}, \dots \right).$$

Show that $T$ is bounded and determined $\|T\|$. Is $T$ injective? Is $T$ surjective?

11. Consider the *Volterra operator* $Vf(x) = \int_0^x f(t)\, \mathrm{d}\, t$ on $L^2[0, 1]$.
   (a) Show that the Volterra operator is compact.
   (b) Find the adjoint $V^*$. It is true that $V$ is selfadjoint?
   (c) Show that its spectrum is $\{0\}$, and that it has no eigenvalues.

12. Let $T \in \mathcal{B}(\mathcal{H})$ be arbitrary. Prove that
   (a) $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$.
   (b) $\lambda \in \sigma_p(T)$ iff $\mathrm{Ran}(\bar{\lambda} I - T)$ is not dense.

13. Show that the residual spectrum of a normal operator $T \in \mathcal{B}(\mathcal{H})$ is always empty.

14. Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Prove the following statements.
   (a) $\|T^2\| = \|T\|^2$.
   (b) $\|T^{2^k}\| = \|T\|^{2^k}$ for every positive integer $k$.
   (c) For the spectral radius of $T$, $r(T) = \|T\|$.
   (d) $\|T^n\| = \|T\|^n$ for every positive integer $n$.

15. What is the set of eigenvalues for the left shift operator

$$S_L : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$$

(a) as an $\mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ operator?

(b) as an $\ell_\infty \to \ell_\infty$ operator?

(c) as an $\ell_p \to \ell_p$ operator?

16. Consider the left shift operator $S_L$ and the right shift operator

$$S_R : (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, x_3, x_4, \dots)$$

as $\ell_2 \to \ell_2$ operators. Determine their adjoints and find their point-, continous- and residual spectrum. What can we say, when we consider $S_L$ and $S_R$ as $\ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ operators?

17. Let $X$ be a Banach space, $A \in \mathcal{B}(X)$, $\lambda \in \mathbb{C}$, and assume there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ so that $\|x_n\| = 1$ and $Ax_n - \lambda x_n \to 0$ as $n \to \infty$. Prove that $\lambda \in \sigma(A)$.

18. Let $T \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator and $\alpha \in \mathbb{C}$ with $\Im\alpha \neq 0$. Prove that the operator

$$U = (\bar{\alpha}I + T)(\alpha + T)^{-1}$$

is unitary.

19. Let $T \in \mathcal{B}(\mathcal{H})$. Prove that $\operatorname{Ker}(T^*T) = \operatorname{Ker} T$.

20. Let $T, S \in \mathcal{B}(\mathcal{H})$. Prove the Resolvent identities, i.e.

(a) Let $\mu, \lambda \in \rho(T)$, then

$$R_\lambda(T) - R_\mu(T) = (\lambda - \mu)R_\lambda(T)R_\mu(T).$$

(b) For $\lambda \in \rho(T) \cap \rho(S)$

$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(T - S)R_\lambda(S).$$

21. Consider $C[0,1]$ with the supremum norm and let define the following operator

$$T : C[0,1] \to C[0,1], \quad (Tf)(x) = xf(x).$$

Determine the spectrum of $T$ (point, continous and residual).